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Boolean Topological Distributive Lattices and Canonical Extensions

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Abstract This paper presents a unified account of a number of dual category equivalences of relevance to the theory of canonical extensions of distributive lattices. Each of the categories involved is generated by an object having a two-element underlying set; additional structure may be algebraic (lattice or complete lattice operations) or relational (order) and, in either case, topology may or may not be included. Among the dualities considered is that due to B. Banaschewski between the categories of Boolean topological bounded distributive lattices and the category of ordered sets. By combining these dualities we obtain new insights into canonical extensions of distributive lattices.

Keywords Topological lattice · Priestley duality · Canonical extension · Profinite completion

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1 Introduction

The purpose of this paper is to provide a fresh perspective on canonical extensions of distributive lattices by studying several natural categories of structures and the connections between them. What these categories have in common is that each is generated by an object based on the two-element set {0, 1}, viewed as an ordered set, a lattice, or a complete lattice, and with or without the discrete topology. Many of these categories have undergone, over a period of about 70 years, a number of reincarnations. Their appearance, in different guises and playing a variety of roles, is indicative of the interest that they have provoked from researchers in diverse fields. The present reappraisal has been prompted by developments in the theory of canonical extensions, specifically for distributive lattice expansions (DLEs); by a DLE we mean an algebra that is a distributive lattice with additional operations. In the course of our review we are able to fit a duality first obtained by Banaschewski [2] into a general scheme. Banaschewski's duality was later revisited by Choe [7], who derived it by a method different from Banaschewski's, but it still seems to be less well known than it deserves to be.[1]

To set this paper in context, let us begin with a brief historical review. Canonical extensions, for Boolean algebras with operators (BAOs), originated in the classic work of Jónsson and Tarski [37]. Their work was principally motivated by their interest in relation algebras, and the import of their work for the relational (Kripke frame) semantics of modal and other logics has only been fully recognised relatively recently; see, for example, [5], pp. 41, 328 and also Jónsson's comments in his survey article [36], p. 245. In the same article (see p. 211), Jónsson makes explicit the connection between canonical extensions for Boolean algebras and M. H. Stone's topological duality: one may view canonical extensions as providing a way to cast Stone duality for Boolean algebras in purely algebraic terms. Here it should be remembered that the introduction of topological methods into algebra was not universally embraced: Stone's maxim "one must always topologize" was, prior to the 1970s, at worst antipathetic and at best unnatural to some of those studying lattice-ordered algebraic structures. The restriction to algebras with a Boolean algebra reduct, rather than on those having a distributive lattice reduct persisted until the mid 1990s. Then Jónsson and Gehrke published a series of papers, culminating in [25], showing that the canonical extensions methodology could be extended very successfully to the distributive-lattice setting and so greatly extending the range of logics to which it could be applied. It was also shown by Gehrke and Harding [23] that the theory of Galois connections could be exploited to define, and abstractly to characterise, canonical extensions of (bounded) lattices. Finally, an extension to poset expansions has been developed by Dunn, Gehrke and Palmigiano [21]. However, since the variety of distributive lattices is generated by a single finite algebra whereas the variety of lattices is not, it is not surprising that the distributive case yields a richer theory, at least as regards its connections, through various

[1] Since the submission of the first version of this paper we have learned that the Banaschewski duality, and its relationship to Priestley duality, was also found independently in the 1990s by Gehrke (unpublished manuscript).

dualities, with other structures. We shall henceforth in this paper work exclusively with lattices that are distributive. On the other hand, the presence or absence of universal bounds for these lattices is not a major issue, since the theory can be adapted to accommodate either situation.

From the point of view of logic, the significance of canonical extensions is that they provide a natural route to complete relational semantics for many propositional logics; see for example [22, 26, 44]. A variety $\mathcal{V}$ of algebraic models, usually obtained as the Lindenbaum–Tarski algebras of a logic $\mathcal{L}$, is considered. A class $\mathcal{V}^+$ of *concrete* algebras is then constructed so that each $A \in \mathcal{V}$ embeds in an algebra $A^\sigma \in \mathcal{V}^+$; under quite weak assumptions, this leads to a covariant functor from $\mathcal{V}$ into $\mathcal{V}^+$. The variety $\mathcal{V}$ is said to be *canonical* if $A^\sigma \in \mathcal{V}$, for all $A \in \mathcal{V}$, that is, if the algebraic laws of $\mathcal{V}$ are preserved under canonical extension. If this is the case, it is possible to move via a contravariant functor from $\mathcal{V}^+$ to a category of relational frames, on which the laws of $\mathcal{V}$ can be captured, in an algorithmic way, by first-order definable properties. By adding topology on the dual (frame) side, it is in many instances possible to capture from the frames not just the class of concrete algebras in $\mathcal{V}$ but also the original variety $\mathcal{V}$. In this situation one has a topological duality for $\mathcal{V}$; this sits on top of Priestley duality [41] for the underlying lattices. Many dualities for varieties of distributive-lattice-based algebras studied for their intrinsic algebraic interest, although originally derived by somewhat ad hoc methods, do in fact fit into this general scheme. In summary, canonical extensions provide, in the setting of canonical varieties of distributive lattice expansions at least, a *uniform* method for deriving dual category equivalences.

We have not specified how the canonical extension of a DLE is to be defined. Traditionally this is done in two stages (we note that [33], which exploits the theory of natural dualities, provides a rare exception to date). At the first stage, an appropriate lattice completion A^σ of the lattice reduct of any $A \in \mathcal{V}$ is defined. One way to obtain A^σ is to take it to be the lattice of all down-sets of the Priestley dual space X of A, where A is (isomorphic to) the lattice of all clopen down-sets of X. (We note that A^σ can in fact be defined constructively, contrary to what the above description might suggest; see Section 4.) This stage of the process, carried out within the category of distributive lattices, works extremely smoothly. It can, in a very natural way, be made functorial, and the (covariant) functor $^\sigma$ is a reflector. At the second stage the non-lattice basic operations are extended to the completion. The way this process is traditionally approached owes much to the subject's roots in modal logic. The unary modality $\Diamond$ preserves 0 and $\vee$ and, dually, $\Box$ preserves 1 and $\wedge$. For a $\Diamond$ operation, the extension can be defined in such a way that it preserves arbitrary joins (likewise, coordinatewise, for multimodal operations), and this is just what is wanted to set up relational frames; see Goldblatt's classic paper [31] or Section 2 of [28]. For operations of other types—and stronger properties may be at least as problematic as weaker ones (see [27], for example)—it is less obvious how the extensions should be defined. However it is usually feasible to do this, and to do it in a way that is (covariantly) functorial. But even when the extended operations are suitably fixed, the algebraic laws holding in the initial variety may fail to hold on canonical extensions. Thus canonicity may be thwarted, but this happens only at the second stage of the two-stage process we have outlined. We draw attention here to an interesting recent paper by Goldblatt [32]; this is framed in terms of modal operators on Boolean algebras, but the ideas are applicable more widely.

We have indicated above that the lattice operations are not treated on a par with the non-lattice operations as regards canonical extensions of DLEs; the former have a distinguished role. Expressed another way, the structures arising as canonical extensions of distributive lattices provide the common platform on which the theory of canonical extensions for all DLEs is built. From a categorical perspective, these structures have quite admirable properties, which do not in general persist once we move from distributive lattices to varieties of DLEs. In addition, structures arising in this manner have, as we indicate below, a multitude of equivalent descriptions and can be characterised in a correspondingly large number of ways, some of which involve topology. They have also arisen in the past in a variety of contexts, most of them far removed from canonical extensions.

Already in the 1950s algebra and topology had been brought together, when work of A. D. Wallace and others initiated the study of topological algebras, and in particular topological lattices (concerning the latter, we note also Papert Strauss's paper [40]). A paper of Numakura [39] revealed a link between topological algebras whose topology is Boolean (that is, compact zero-dimensional) and profiniteness. This area of investigation was subsequently sidelined, when the development of the theory, first of continuous lattices, and later a more general theory of domains, caused interest to shift from topological lattices to topological semilattices. Thus two primary references on the interaction of topology and order, [29] and its successor [30], contain relatively little material overlapping with that we present here. On the other hand, as its list of authors may indicate, the presentation in this paper owes much to the theory of natural dualities for quasi-varieties of algebras, for which [8] provides a comprehensive basic reference. We would like to acknowledge the contribution made by M. Ploščica who with M. Haviar rediscovered Banaschewski's duality in the mid-1990s which later initiated our investigations.

The paper is organised as follows. In the next section we make explicit the isomorphism between two categories central to our investigation. In the following one we use this isomorphism to give a direct proof of Banaschewski's duality between the category of Boolean topological bounded distributive lattices and the category of ordered sets. Our proof follows similar lines to that of Choe [7] but is more overtly lattice-theoretic. We see in particular how this duality may be seen as relating to Priestley duality by a swap of topology from the category on one side of a dual equivalence to that on the other. This process is explored in much greater generality in [11]. Finally, in Section 4, we use the same isomorphism, but in the opposite direction, to show how canonical extensions of distributive lattices can be viewed from a purely topological standpoint and from the standpoint of a profinite completion.

2 Boolean Topologies and Completeness

We shall make use of a number of different structures based on the set $2 := \{0, 1\}$. In particular,

$$\underline{2} := \langle \{0, 1\}; \vee, \wedge \rangle, \quad \underset{\sim}{2} := \langle \{0, 1\}; \leqslant, 0, 1 \rangle \quad \text{and} \quad 2_{\mathcal{T}} := \langle \{0, 1\}; \mathcal{T} \rangle$$

denote, respectively, the two-element lattice, the two-element bounded ordered set with $0 < 1$ and the two-element discrete space. The topological structures $\underline{2}_{\mathcal{T}}$ and $\underset{\sim}{2}_{\mathcal{T}}$ are obtained by adding the discrete topology $\mathcal{T}$ to $\underline{2}$ and $\underset{\sim}{2}$, respectively.

A *Boolean topological lattice* is a lattice whose underlying set is endowed with a Boolean topology with respect to which the binary operations $\vee$ and $\wedge$ are continuous. (Throughout the paper, in the expression 'Boolean topological lattice,' the adjective 'Boolean' refers to the topological structure rather than the lattice structure.) The following lemma of Numakura shows that Boolean topological distributive lattices arise as closed sublattices of powers of $\underline{2}_{\mathcal{T}}$. (It also follows from more general results on the axiomatisation of topological quasi-varieties proved by Clark et al. [9, Example 5.1].)

Lemma 2.1 (Numakura [39, Theorem 2]) *Every closed sublattice of a power of $\underline{2}_{\mathcal{T}}$ is a Boolean topological distributive lattice. Conversely, every Boolean topological distributive lattice is isomorphic to a closed sublattice of a power of $\underline{2}_{\mathcal{T}}$.*

A non-empty subset K of a complete lattice L is called a *complete sublattice* of L if it is closed under joins and meets (taken in L) of arbitrary *non-empty* subsets. If, in addition, $0_L \in K$ and $1_L \in K$, then we say that K is a *complete* 0,1-*sublattice* of L. Let A be a subset of 2^S, for some set S. The topological closure of A in $2^S_{\mathcal{T}}$ will be denoted by $\overline{A}$. Throughout this paper, we will use the elementary fact that, for all $x \in 2^S$, we have $x \in \overline{A}$ if and only if x is *locally in* A, that is, for every finite subset T of S, there exists $a \in A$ with $x{\restriction}_T = a{\restriction}_T$. We shall write $J \Subset S$ to indicate that J is a finite subset of S, and the constant maps in 2^S onto 0 and 1 will be denoted by $\widehat{0}$ and $\widehat{1}$, respectively.

Lemma 2.2 *Let L be a sublattice of a Boolean topological distributive lattice X.*

(a) *The underlying lattice of X is complete.*
(b) *For all $x \in X$, the following are equivalent* (*with the joins and meets calculated in X*):
    (1) $x \in \overline{L}$;
    (2) $x = \bigvee\{\bigwedge A_i \mid i \in I\}$, *for some non-empty set I and non-empty subsets A_i of L;*
    (3) $x = \bigwedge\{\bigvee A_i \mid i \in I\}$, *for some non-empty set I and non-empty subsets A_i of L.*
(c) *$\overline{L}$ is the complete sublattice of X generated by L.*
(d) *L is a closed sublattice of X if and only if L is a complete sublattice of X.*
(e) *The closure $\overline{A}$ of a filter A in X is $\uparrow \bigwedge A$, and dually for an ideal.*

Proof By Lemma 2.1, we may assume that X is a closed sublattice of $2^S_{\mathcal{T}}$, for some non-empty set S. We begin by proving that $\overline{L}$ is a complete sublattice of $\underline{2}^S$. This will follow easily once we prove that:

$$\varnothing \neq A \subseteq L \implies \bigvee A \in \overline{L} \ \& \ \bigwedge A \in \overline{L}. \qquad (*)$$

Let A be a non-empty subset of L. We shall prove that $x := \bigwedge A$ is locally in L. Let $T \Subset S$ and define

$$T_0 := \{\, s \in T \mid x(s) = 0 \,\} \quad \text{and} \quad T_1 := \{\, s \in T \mid x(s) = 1 \,\}.$$

Assume that T_0 is non-empty, and choose $s \in T_0$. Since $x = \bigwedge A$, there exists $a_s \in A$ with $a_s(s) = 0$. Define $b := \bigwedge\{\, a_s \mid s \in T_0 \,\}$. Clearly, $b \in L$ and $x{\restriction_{T_0}} = \widehat{0}{\restriction_{T_0}} = b{\restriction_{T_0}}$. As $b \geqslant x$, we also have $x{\restriction_{T_1}} = b{\restriction_{T_1}}$, and so $x{\restriction_T} = b{\restriction_T}$. If T_0 is empty, then $x{\restriction_T} = \widehat{1}{\restriction_T}$. So, if we choose b to be any element of A, we have $b \geqslant x$ and hence $x{\restriction_T} = b{\restriction_T}$. Therefore $\bigwedge A$ is locally in L. Dually, $\bigvee A$ is also locally in L.

It is an easy exercise to prove that the closure of a subalgebra of a topological algebra is again a subalgebra. Thus, $\overline{L}$ is a closed sublattice of $\underset{\sim}{2}^S_{\mathcal{T}}$. By applying Eq. $*$ to X and $\overline{L}$ in turn we conclude at once that X is a complete sublattice of $\underset{\sim}{2}^S$ and that $\overline{L}$ is a complete sublattice of X. Thus (a) holds.

We now show that (b) holds. To prove (1) $\Rightarrow$ (2), assume that $x \in \overline{L}$. If $x = \widehat{0}$, then, since x is locally in L, for each $J \Subset S$ there exists $a_J \in L$ with $a_J(s) = 0$, for all $s \in J$. Thus $x = \bigwedge\{\, a_J \mid J \Subset S \,\}$, as required. By duality, we may assume that $x \notin \{\widehat{0}, \widehat{1}\}$. Hence, the sets

$$S_0 := \{\, s \in S \mid x(s) = 0 \,\} \quad \text{and} \quad S_1 := \{\, s \in S \mid x(s) = 1 \,\}$$

are non-empty. Let $J \Subset S_0$ and $K \Subset S_1$. As x is locally in L, there exists $a_{J,K} \in L$ with $a_{J,K}(s) = 0$, for all $s \in J$, and $a_{J,K}(s) = 1$, for all $s \in K$. A simple calculation shows that

$$x = \bigvee \left\{ \bigwedge \{\, a_{J,K} \mid J \Subset S_0 \,\} \,\middle|\, K \Subset S_1 \right\},$$

as required. By duality, we also have (1) $\Rightarrow$ (3). Since $\overline{L}$ is a complete sublattice of $\underset{\sim}{2}^S$ containing L, the remaining implications, (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1), follow at once.

Parts (c) and (d) follow easily from (b). Finally, consider (e). Note that, by (b), applied with $L = A$, and the fact that A is an up-set, we have that

$$\overline{A} = \left\{ \bigwedge B \,\middle|\, \emptyset \neq B \subseteq A \right\}.$$

It follows immediately that $\overline{A} \subseteq {\uparrow} \bigwedge A$. In the other direction, take $x \geqslant \bigwedge A$. By Lemma 2.1 and (d), we have that $\vee$ in X distributes over $\bigwedge$. Hence

$$x = x \vee \bigwedge A = \bigwedge \{\, x \vee a \mid a \in A \,\}.$$

Since $x \vee a \in A$, for each $a \in A$, we have $x \in \overline{A}$. □

Lemma 2.3 *Let L be a Boolean topological distributive lattice. Then the topology on L agrees with the interval topology, that is, $\mathcal{F} := \{\, {\uparrow}a \mid a \in L \,\} \cup \{\, {\downarrow}a \mid a \in L \,\}$ is a subbasis for the closed subsets of L.*

Proof Let $a \in L$. Since ${\uparrow}a$ is a complete sublattice of L, it is topologically closed in L, by Lemma 2.2(d). Dually, ${\downarrow}a$ is closed in L. Thus, $\mathcal{F}$ is a family of closed subsets of L.

By Lemma 2.1, we may assume that L is a closed sublattice of $\underset{\sim}{2}^S_{\mathcal{T}}$, for some set S. Let $s \in S$, let $i \in \{0, 1\}$ and consider the set $U_{s,i} := \{\, a \in L \mid a(s) = i \,\}$. Since L is closed under pointwise joins, $U_{s,0}$ has a largest element, say a_s, and $U_{s,0} = {\downarrow}a_s$. Thus, $U_{s,0} \in \mathcal{F}$. Similarly, $U_{s,1} \in \mathcal{F}$. By the definition of the product topology, the family of all non-empty subsets of L of the form $U_{s,i}$, for $s \in S$ and $i \in \{0, 1\}$, is a sub-basis for the closed subsets of L, whence the result follows. □

The following lemma, which is an immediate corollary of the previous one, complements Lemma 2.1.

Lemma 2.4 *Let L be a lattice isomorphic to a complete sublattice of a power of $\underline{2}$. When endowed with the interval topology, L becomes a Boolean topological distributive lattice.*

By Lemma 2.2(d), working with topologically closed sublattices of powers of $\underline{2}_{\mathcal{T}}$ is equivalent to working with complete sublattices of powers of $\underline{2}$. The following often-rediscovered characterisation of such complete lattices dates in part to the early 1950s, when Raney [42, 43], Balacharandran [1] and Büchi [6] each identified some of the equivalences. More comprehensive lists of equivalences were recorded by Davey [10] and Erné [16] (cf. also Hofmann and Mislove [34]). For a proof of the theorem (and the missing definitions) see Theorem 10.29 of Davey and Priestley [12]. No fully satisfactory name has ever been devised for the members of the important class of lattices the theorem describes. However we note that the term *superalgebraic* is quite commonly used in the more recent literature.

Theorem 2.5 *Let L be a lattice. The following are equivalent*:

(1) *L is isomorphic to a complete sublattice a power of $\underline{2}^S$*;
(2) *L is isomorphic to a complete lattice of sets*;
(3) *L is isomorphic to the lattice $\mathcal{O}(P)$ of all down-sets of some ordered set P*;
(4) *L is algebraic and completely distributive*;
(5) *L is distributive and doubly algebraic* (*i.e. both algebraic and dually algebraic*);
(6) *L is algebraic, distributive and every element of L is a join of completely join-irreducible elements of L*;
(7) *L is complete, satisfies the join-infinite distributive law and every element of L is a join of completely join-irreducible elements of L*;
(8) *L is complete and every element of L is a join of completely join-prime elements of L*;
(9) *the map $\mu_L : L \to \mathcal{O}(\mathcal{J}^\infty(L))$, from L to the lattice of down-sets of the ordered set of completely join-irreducible elements of L, is an isomorphism.*

We remark that much of the content of this section can be seen as related to, or specialising, more general results concerning continuous and bicontinuous lattices. In the latter, the unit interval replaces $\underline{2}_{\mathcal{T}}$, and compact Hausdorff spaces replace Boolean ones. Powers of the unit interval (and hence also powers of $\underline{2}_{\mathcal{T}}$ and their closed sublattices), are completely distributive, and hence a fortiori complete and bicontinuous (cf. Papert Strauss [40]). The portmanteau result, Proposition VII-2.10 [30, Section VII-2], gives a list of nine equivalent characterisations of what are known as linked bicontinuous lattices. This can be specialised from the setting of continuous lattices to that of algebraic ones essentially giving Theorem 2.5 as a corollary. The proof given in [30] draws on much of the theory of continuous lattices and domains presented elsewhere in the book. (We note that the corresponding proposition in [29], Proposition VII-2.9, presents five equivalent conditions rather than nine.)

Concerning the wider context for Lemma 2.3, we recall that the bicontinuous lattices are precisely the order-topological ones. That is, they are precisely those lattices whose order convergence is topological and makes the lattice operations continuous (cf. Erné [14]). From this it follows that the adherence points of a subset are the liminfs and limsups of filterbases on that subset. The limits of ultrafilters relative to order convergence are just the unique limits relative to the interval topology; hence, a lattice is complete and its interval topology is Hausdorff if and only if every ultrafilter order converges (cf. Erné and Weck [20]). We do not make use of these facts subsequently.

Let $\mathfrak{L}$ be the category whose objects are Boolean topological lattices and whose morphisms are continuous lattice homomorphisms, and let $\mathfrak{C}$ be the category whose objects are doubly algebraic distributive lattices and whose morphisms are complete lattice homomorphisms (that is, maps that preserves joins and meets of arbitrary *non-empty* subsets). We have already seen that $\mathfrak{L}$ and $\mathfrak{C}$ are isomorphic at the object level. We now show that this extends to morphisms. The following lemma is essentially well known. Indeed, it is known that a map between ordered sets preserves up-directed joins and down-directed meets if and only if it is continuous with respect to any topology between the interval topology and the order topology (cf. Erné and Gatzke [19]). For completeness we include a direct proof based on the preceding lemmas.

Lemma 2.6 *Let $L, M \in \mathfrak{L}$ and let $f : L \to M$ be a lattice homomorphism. Then f is continuous if and only if f is a complete lattice homomorphism.*

Proof First, assume that f is continuous. Let $\varnothing \neq A \subseteq L$. We shall prove that $f(\bigwedge A) = \bigwedge f(A)$. Since f is order-preserving, we have $f(\bigwedge A) \leqslant \bigwedge f(A)$. The set $\uparrow\bigwedge f(A)$ is closed in M, by Lemma 2.3, and hence $B := f^{-1}(\uparrow\bigwedge f(A))$ is closed in L. Clearly, B is a sublattice of L and $A \subseteq B$. By Lemma 2.2(c), we have $\bigwedge A \in B$, whence $f(\bigwedge A) \in \uparrow\bigwedge f(A)$. Consequently, f preserves meets of arbitrary non-empty sets. The proof for joins is dual.

Conversely, suppose that f is a complete lattice homomorphism. Then, for all $a \in M$, we have that $f^{-1}(\uparrow a)$ is either empty or equal to $\uparrow\bigwedge f^{-1}(\uparrow a)$ and $f^{-1}(\downarrow a)$ is either empty or equal to $\downarrow\bigvee f^{-1}(\downarrow a)$. It follows immediately from Lemma 2.3 that f is continuous. □

We have established the following basic result.

Theorem 2.7 *The categories $\mathfrak{L}$ of Boolean topological distributive lattices and $\mathfrak{C}$ of doubly algebraic distributive lattices are isomorphic.*

It follows almost immediately from this result that the category $\mathfrak{L}_{01}$ of Boolean topological distributive lattices with continuous 0,1-lattice homomorphisms and the category $\mathfrak{C}_{01}$ of doubly algebraic distributive lattices with complete 0,1-lattice homomorphisms are also isomorphic.

3 Banaschewski's Duality for $\mathfrak{L}$

In 1976, Banaschewski [2] proved that there is a full duality (that is, a dual category equivalence) between the category $\mathfrak{L}_{01}$ of Boolean topological distributive lattices (with continuous 0,1-preserving lattice homomorphisms) and the category $\mathfrak{P}$ of ordered sets. Although Banaschewski's original paper is difficult to obtain, the duality can be found in Johnstone's text [35], where it is established by methods that are more general than those in [2] and similarly indirect. A more direct, essentially topological, approach is taken by Choe in [7]. We now sketch a proof of the corresponding full duality for the category $\mathfrak{L}$ of Boolean topological distributive lattices. This approach is closely related to Choe's, but replaces the category $\mathfrak{L}$ by its isomorphic copy $\mathfrak{C}$. This replacement has the advantage of showing very clearly how the result extends the corresponding one for the finite case and reduces the proof to completely straightforward lattice theory mimicing that required for finite objects.

Recall that $\underset{\sim}{2} = \langle\{0,1\}; \leqslant, 0, 1\rangle$ denotes the two-element bounded ordered set with $0 < 1$ and with 0 and 1 as nullary operations. The quasi-variety $\mathbb{ISP}(\underset{\sim}{2})$ of all structures of the same type as $\underset{\sim}{2}$ that are isomorphic to substructures of powers of $\underset{\sim}{2}$ is obviously the class of all bounded ordered sets. Let $\mathfrak{P}_{01}$ be the category whose objects are *non-trivial* bounded ordered sets and whose morphisms are order-preserving, 0,1-preserving maps.

There are natural contravariant hom-functors $D = \mathfrak{L}(-, \underline{2}_{\mathfrak{T}}) : \mathfrak{L} \to \mathfrak{P}_{01}$ and $E = \mathfrak{P}_{01}(-, \underset{\sim}{2}) : \mathfrak{P}_{01} \to \mathfrak{L}$. For each object L of $\mathfrak{L}$, we define

$$D(L) := \langle \mathfrak{L}(L, \underline{2}_{\mathfrak{T}}); \leqslant, 0, 1\rangle \;\leqslant\; \underset{\sim}{2}^{L},$$

that is, the order and constants 0 and 1 on $D(L)$ are inherited from the pointwise structure on $\underset{\sim}{2}^{L}$. The functor D is defined on morphisms in the usual way via composition: for $u \in \mathfrak{L}(L, K)$, define $D(u) : D(K) \to D(L)$ by $D(u)(x) := x \circ u$, for all $x \in \mathfrak{L}(K, \underline{2}_{\mathfrak{T}})$. For each object P of $\mathfrak{P}$, we define

$$E(P) := \langle \mathfrak{P}_{01}(P, \underset{\sim}{2}); \vee, \wedge, \mathfrak{T}\rangle \;\leqslant\; \underline{2}_{\mathfrak{T}}^{P},$$

that is, $E(P)$ inherits its operations from the pointwise operations on $\underline{2}^{P}$ and its topology from the product topology on $2_{\mathfrak{T}}^{P}$. It is a very easy exercise to see that the homset $\mathfrak{P}_{01}(P, \underset{\sim}{2})$ is a closed subset of $2_{\mathfrak{T}}^{P}$, whence $E(P)$ belongs to $\mathfrak{L}$. Again, E is defined on morphisms in the usual way via composition: for $\varphi \in \mathfrak{P}_{01}(P, Q)$, define $E(\varphi) : E(Q) \to E(P)$ by $E(\varphi)(\alpha) := \alpha \circ \varphi$, for all $\alpha \in \mathfrak{P}_{01}(Q, \underset{\sim}{2})$. It is very easy to see that $E(\varphi)$ preserves $\vee$ and $\wedge$. In fact $E(\varphi)$ preserves arbitrary non-empty joins and meets, because these are defined pointwise and the codomain is finite. So $E(\varphi)$ is a $\mathfrak{C}$-morphism. It is equally easy to show directly that $E(\varphi)$ is continuous, using the fact that the subbasic open sets in $E(P)$ are of the form $U_{a,i} := \{\, \alpha \in \mathfrak{P}_{01}(P, \underset{\sim}{2}) \mid \alpha(a) = i\,\}$, with $a \in P$ and $i \in \{0, 1\}$.

As in the case of the Birkhoff duality for finite distributive lattices [4], there are alternative internal descriptions of these functors.

Lemma 3.1 *Let $L \in \mathfrak{L}$ and let $P \in \mathfrak{P}_{01}$. Then $D(L)$ is dually order-isomorphic to the ordered set $\mathcal{J}^{\infty}(L) \dot\cup \{0, 1\}$ of completely join-irreducible elements of L with new bounds adjoined, and $E(P)$ is dually order-isomorphic (and therefore dually isomorphic as a complete lattice) to the lattice $\mathcal{O}(P\backslash\{0, 1\})$ of down-sets of P with its bounds removed.*

Proof Begin by replacing the category $\mathfrak{L}$ by the isomorphic category $\mathfrak{C}$. Now let $L \in \mathfrak{C}$ and let $P \in \mathfrak{P}_{01}$. A non-constant map $x : L \to 2$ is a complete lattice homomorphism if and only if it is the characteristic function of $\uparrow a$ for some completely join-prime element $a \in L$. Since L satisfies the join-infinite distributive law (by Theorem 2.5 above), $a \in L$ is completely join-prime if and only if it is completely join-irreducible. Thus, $\mathfrak{C}(L, \underset{\sim}{2})$ is dually order-isomorphic to the ordered set $\mathcal{J}^\infty(L) \dot\cup \{0, 1\}$. Similarly, a map $\alpha : P \to 2$ is order-preserving if and only if it is the characteristic function of $P \backslash A$, for some down-set A of P. Thus, $\mathfrak{P}_{01}(P, \underset{\sim}{2})$ is dually order-isomorphic to the lattice $\mathcal{O}^*(P)$ of non-empty proper down-sets of P, which, since P is non-trivial, is order-isomorphic to the lattice $\mathcal{O}(P\backslash\{0, 1\})$ of down-sets of P with its bounds removed. □

For every $L \in \mathfrak{L}$ and $P \in \mathfrak{P}_{01}$, we define the evaluation maps $\eta_L : L \to ED(L)$ and $\varepsilon_P : P \to DE(P)$ in the usual way. For $a \in L$, $x \in D(L)$ and $p \in P$, $\alpha \in E(P)$, we define

$$\eta_L(a)(x) := x(a) \quad \text{and} \quad \varepsilon_P(p)(\alpha) := \alpha(p).$$

It is easy to verify that η_L and ε_P are morphisms in the categories $\mathfrak{L}$ and $\mathfrak{P}_{01}$, respectively.

Lemma 3.2 *The maps η_L and ε_P are isomorphisms, for all $L \in \mathfrak{L}$ and $P \in \mathfrak{P}_{01}$.*

Proof Again, we replace the category $\mathfrak{L}$ by the isomorphic category $\mathfrak{C}$. With the various characterisations of the objects in $\mathfrak{C}$ provided by Theorem 2.5, and with Lemma 3.1 in hand, the proof is a simple modification of the proof in the finite case as given, for example, in Davey and Priestley [12]: replace join-irreducible and join-prime elements by completely join-irreducible and completely join-prime elements, and replace applications of the distributive law by the join-infinite distributive law. Below we indicate the steps involved.

Under the correspondence provided by Lemma 3.1, the map $\eta_L : L \to ED(L)$ corresponds to the map $\mu_L : L \to \mathcal{O}(\mathcal{J}^\infty(L))$ defined in (9) of Theorem 2.5. Thus, since $L \in \mathfrak{C}$, we are done by (1) $\Rightarrow$ (9) of Theorem 2.5. (A direct proof that μ_L is an isomorphism follows the lines of the proof of Theorem 5.12 in [12].) Similarly, the map $\varepsilon_P : P \to DE(P)$ corresponds to the map $\xi_P : p \mapsto \downarrow p$ from $P \setminus \{0, 1\}$ to the ordered set $\mathcal{J}^\infty(\mathcal{O}(P\backslash\{0, 1\}))$. The proof that ξ_P is an order-isomorphism is easy and follows the lines of the proof of Theorem 5.9 in [12]. □

Very easy calculations, analogous to those in [13, proof of 1.5], show that both $\eta : \mathrm{id}_{\mathfrak{L}} \to ED$ and $\varepsilon : \mathrm{id}_{\mathfrak{P}_{01}} \to DE$ are natural transformations. Thus, we have proved a variant of Banaschewski's duality for the category of Boolean topological distributive lattices. Completely trivial modifications to these arguments yield a direct proof of Banaschewski's original duality between the categories $\mathfrak{L}_{01}$ and $\mathfrak{P}$.

Theorem 3.3 (Banaschewski [2]) *The functors D and E establish a full duality between the categories $\mathfrak{L}$ and $\mathfrak{P}_{01}$, and therefore between the categories $\mathfrak{C}$ and $\mathfrak{P}_{01}$.*

We remark that the duality between $\mathfrak{P}$ and $\mathfrak{L}_{01}$ and its variants can be obtained as special instances of a wealth of Stone-type dualities (see, for example, Erné [15, 17, 18]).

Banaschewski's duality is very closely related to the corresponding Priestley duality between distributive lattices and bounded Priestley spaces [41]. As before, let $\underline{2} = \langle\{0, 1\}; \vee, \wedge\rangle$ and $\underset{\sim}{2}_{\mathcal{T}} = \langle\{0, 1\}; \leqslant, 0, 1, \mathcal{T}\rangle$ be, respectively, the two-element lattice and the two-element bounded ordered set with the discrete topology. Then $\mathbf{\mathfrak{D}} := \mathbb{ISP}(\underline{2})$ is the variety of distributive lattices and the class $\mathbf{\mathfrak{X}}_{01} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{2}_{\mathcal{T}})$, consisting of all isomorphic copies of substructures of *non-zero* powers of $\underset{\sim}{2}_{\mathcal{T}}$, is the class of all *non-trivial* bounded Priestley spaces. The duality between $\mathbf{\mathfrak{D}}$ and $\mathbf{\mathfrak{X}}_{01}$ is given by the contravariant hom-functors $H = \mathbf{\mathfrak{D}}(-, \underline{2}) : \mathbf{\mathfrak{D}} \to \mathbf{\mathfrak{X}}_{01}$ and $K = \mathbf{\mathfrak{X}}_{01}(-, \underset{\sim}{2}_{\mathcal{T}}) : \mathbf{\mathfrak{X}}_{01} \to \mathbf{\mathfrak{L}}$ (see Clark and Davey [8, 4.3.2] for a brief proof in the setting of natural dualities). Thus, Banaschewski's duality can be thought of as coming from Priestley duality for distributive lattices by swapping the discrete topology $\mathcal{T}$ from the relational side to the algebraic side. Indeed, this is essentially how Banaschewski obtained his duality.

We remark that, for very many varieties of DLEs, nullary operations 0 and 1 are included in the type. In such situations we would expect to want to have 0 and 1 on the algebraic side rather than the relational one (cf. [13, Section 2.8] or [8, Section 4.1]). Consequently we would want to work with $\mathbf{\mathfrak{P}}$ rather than with $\mathbf{\mathfrak{P}}_{01}$. However where non-lattice operations of arity greater than one are present it turns out to be necessary to augment the ordered sets to be used as frames (in the sense the term is used in Kripke-style relational semantics) by adding universal bounds even when 0 and 1 are operations in the algebraic type; see Martínez [38] for an early recognition of this awkwardness and also [27]. There are also important varieties of DLEs – the variety of lattice-ordered groups being an obvious example – in which the underlying lattices are not bounded. For these reasons we have chosen to put 0 and 1 on the relational side in our presentation in this section and the preceding one.

4 Boolean Topological Distributive Lattices and Canonical Extensions

The breakthrough whereby the theory of canonical extensions was extended from the setting of BAOs to that of distributive lattices with operators was made by Gehrke and Jónsson in [24]. In that paper the construction was explicit, and based on the dual space of the underlying lattice. Subsequently an abstract definition and characterisation of the canonical extension was found. Gehrke and Harding [23] proved that every bounded lattice L has a canonical extension and that any two canonical extensions of L are isomorphic via an isomorphism that fixes the elements of L. (Earlier, though published later, Gehrke and Jónsson [25] proved the corresponding facts in the distributive case.) In this section we exploit the abstract characterisation to give equivalent topological formulations of the conditions involved and thereby are able to give a number of new descriptions of those complete lattices that, up to isomorphism, occur as canonical extensions of distributive lattices.

We begin by recalling briefly the necessary definitions. Let L be a sublattice of a complete lattice C. Then C is called a *completion* of L. (More generally, if $e : L \to C$ is an embedding of the lattice L into the complete lattice C, then the pair (e, C) is also called a *completion* of L.) The completion C of L is said to be *dense* if every element of C can be expressed both as a join of meets and as a meet of joins of elements of L, and C is called a *compact* completion of L if, for all non-empty subsets A and B of L, we have $\bigwedge A \leqslant \bigvee B$ implies $\bigwedge A_0 \leqslant \bigvee B_0$, for some $A_0 \Subset A$ and $B_0 \Subset B$, or equivalently, if for every filter A of L and every ideal B of L, we have $\bigwedge A \leqslant \bigvee B$

implies $A \cap B \neq \varnothing$. A *canonical extension* of a lattice L is a completion C of L that is both dense and compact. As noted above, any bounded lattice possesses a canonical extension, and this is unique up to isomorphism. It is easy to extend this result to not-necessarily bounded lattices.

We now give an interpretation of the canonical extension of a distributive lattice in terms of Boolean topological distributive lattices. Let A and B be respectively a filter and an ideal in a lattice L with A and B disjoint. Then (A, B) is called a *filter-ideal pair*.

Theorem 4.1 *Let X be a Boolean topological distributive lattice and let L be a sublattice of X.*

(a) *The (underlying lattice of) X is a completion of L.*
(b) *X is a dense completion of L if and only if L is topologically dense in X.*
(c) *X is a compact completion of L if and only if, for any filter-ideal pair (A, B), the topological closures of A and B in X are disjoint.*

Proof Parts (a) and (b) are an immediate consequence of Lemma 2.2. We now prove (c). Assume that X is a compact completion of L. Let A be a filter of L and B be an ideal of L with $\overline{A} \cap \overline{B} \neq \varnothing$. Then Lemma 2.2(e) (applied to X) tells us that $\uparrow_X(\bigwedge A) \cap \downarrow_X(\bigvee B) \neq \varnothing$ and so $\bigwedge A \leqslant \bigvee B$. As X is a compact completion of L, there exists $A_0 \Subset A$ and $B_0 \Subset B$ with $\bigwedge A_0 \leqslant \bigvee B_0$. But $\bigwedge A_0 \in A$ and $\bigvee B_0 \in B$, and consequently $A \cap B \neq \varnothing$.

Conversely, assume that the members of any filter-ideal pair have disjoint closures. Let $A, B \subseteq L$ with $\bigwedge A \leqslant \bigvee B$. Let C be the filter of L generated by A and let D be the ideal of L generated by B. By Lemma 2.2, $\bigwedge A \in \overline{C}$ and $\bigvee B \in \overline{D}$. Hence $\overline{C} \cap \overline{D} \neq \varnothing$ (note $\overline{C}$ is an up-set and $\overline{D}$ is a down-set, by Lemma 2.2(e)). Thus, $C \cap D \neq \varnothing$, by assumption, and so there exists $A_0 \Subset A$ and $B_0 \Subset B$ with $\bigwedge A_0 \leqslant \bigvee B_0$. Thus X is a compact completion of L. □

Assume that L is a sublattice of $\underline{2}^Z$, for some set Z. Then the above theorem indicates that the topological closure of L in $\underline{2}^Z_{\mathcal{T}}$ is a natural candidate for a canonical extension of L. Our next theorem shows that $\overline{L}$ is a compact extension, and therefore a canonical extension, of L if and only if a certain topology on the exponent Z is compact. (Note that we do not require that compact spaces be Hausdorff, that is, in our usage compact spaces are those which are called quasicompact in [29, 30].)

Theorem 4.2 *Let L be a sublattice of $\underline{2}^Z$, for some set Z.*

(a) *For all $A, B \subseteq L$,*

$$\bigwedge A \leqslant \bigvee B \iff \bigcup_{a \in A} a^{-1}(0) \cup \bigcup_{b \in B} b^{-1}(1) = Z.$$

(b) *The following are equivalent*:

  (1) *the lattice $\underline{2}^Z$ is a compact completion of L*;
  (2) *the topology on Z with subbasis $\{a^{-1}(0) \mid a \in L\} \cup \{b^{-1}(1) \mid b \in L\}$ is compact*;
  (3) *there is a compact topology on Z such that L is a sublattice of the lattice $\mathrm{C}(Z, \underline{2}_{\mathcal{T}})$ of continuous maps from Z into $\underline{2}_{\mathcal{T}}$.*

Proof To prove (a), we establish the contrapositive in each direction.

$$\begin{aligned}\bigwedge A \nleqslant \bigvee B &\iff (\exists z \in Z)\ \Big(\bigwedge A\Big)(z) = 1\ \ \&\ \ \Big(\bigvee B\Big)(z) = 0\\ &\iff (\exists z \in Z)(\forall a \in A)(\forall b \in B)\ a(z) = 1\ \ \&\ \ b(z) = 0\\ &\iff (\exists z \in Z)\ z \notin \bigcup_{a\in A} a^{-1}(0) \cup \bigcup_{b\in B} b^{-1}(1).\end{aligned}$$

We now prove (b). Assume that (1) holds. By Alexander's Subbasis Theorem, to prove (2) it suffices to show that every cover of Z by subbasic open sets has a finite subcover. This follows very easily, via two applications of (a), from the fact that $\underline{2}^Z$ is a compact completion of L. That (2) implies (3) is trivial. That (3) implies (1) is an easy consequence, again using two applications of (a), of the compactness of Z and the continuity of the elements of L. □

Every distributive lattice L can be represented as a sublattice of the lattice $\mathrm{C}(Z, \underline{2}_{\mathcal{T}})$ of continuous maps from a compact space Z into $\underline{2}_{\mathcal{T}}$. For example, first represent L as a sublattice of $\underline{2}^S$, for some set S, equip S with the discrete topology, and then take Z to be the Stone–Čech compactification of S. Alternatively, take Z to be the Priestley dual of L. Thus, our next result, which is an immediate corollary of Theorem 4.1(b) and Theorem 4.2(b), provides us with a ready way to obtain topological structures serving as canonical extensions of a given distributive lattice. Below, $\underline{2}_{\mathcal{T}}^Z$ denotes the Z-fold product of $\underline{2}_{\mathcal{T}}$ (the discretely topologised two-element lattice); the more correct notation $(\underline{2}_{\mathcal{T}})^Z$ would be rather clumsy.

Theorem 4.3 *Assume that Z is a compact topological space and that L is a sublattice of the lattice $\mathrm{C}(Z, \underline{2}_{\mathcal{T}})$ of continuous maps from Z into $\underline{2}_{\mathcal{T}}$. Then the topological closure of L in $\underline{2}_{\mathcal{T}}^Z$ is a canonical extension of L.*

Let L be a distributive lattice. When ordered pointwise, the homset $\mathbf{\mathfrak{D}}(L, \underline{2})$ is a bounded ordered set. Thus $\mathfrak{P}_{01}(\mathfrak{D}(L, \underline{2}), \underset{\sim}{2})$ is a complete sublattice of $\underline{2}^{\mathfrak{D}(L,\underline{2})}$. Accordingly, a clear choice for Z in the previous theorem is (the underlying topological space of) the bounded Priestley space

$$H(L) = \langle \mathfrak{D}(L, \underline{2}); \leqslant, 0, 1, \mathcal{T}\rangle \ \leqslant\ \underset{\sim}{2}_{\mathcal{T}}^L.$$

This suggests a natural description in terms of homsets of a canonical extension of a distributive lattice, akin to that available in the more familiar case of *bounded* distributive lattices. This description can be recast in terms of the up-sets of the ordered set of prime filters of L or alternatively in terms of the down-sets of the ordered set of prime ideals of L. We shall denote the latter set by $\mathcal{I}_{\mathrm{p}}(L)$.

Theorem 4.4 *Let L be a distributive lattice.*

(a) *Define the map $e : L \to \mathfrak{P}_{01}(\mathfrak{D}(L, \underline{2}), \underset{\sim}{2})$ by $e(a)(x) := x(a)$, for all $a \in L$ and all $x \in \mathfrak{D}(L, \underline{2})$. Then $(e, \mathfrak{P}_{01}(\mathfrak{D}(L, \underline{2}), \underset{\sim}{2}))$ is a canonical extension of L.*

(b) *Define the map $e : L \to \mathcal{O}(\mathcal{I}_{\mathrm{p}}(L))$ by $e(a) := \{\, I \in \mathcal{I}_{\mathrm{p}}(L) \mid a \notin I \,\}$. Then $(e, \mathcal{O}(\mathcal{I}_{\mathrm{p}}(L)))$ is a canonical extension of L.*

Proof By the previous theorem, to prove (a) we must show that the topological closure of $e(L)$ in $2_{\mathcal{T}}^{\mathbf{D}(L,\underline{2})}$ is $\mathcal{P}_{01}(\mathbf{D}(L,\underline{2}),\underset{\sim}{2})$. It is enough to prove that every order-preserving, 0,1-preserving map from $\mathbf{D}(L,\underline{2})$ to $\underset{\sim}{2}$ is locally an evaluation. By Priestley duality, the evaluation maps from $\mathbf{D}(L,\underline{2})$ to 2 are precisely the bounded Priestley space morphisms from $H(L)$ to $\underset{\sim}{2}_{\mathcal{T}}$. Thus, it suffices to show that if X is a bounded Priestley space and Y is a finite subset of X, then every order-preserving, 0,1-preserving map from $Y \cup \{0, 1\}$ to $\underset{\sim}{2}$ extends to a continuous, order-preserving, 0,1-preserving map from X to $\underset{\sim}{2}_{\mathcal{T}}$. This is a very easy consequence of the definition of a Priestley space and also follows from the well-known fact that $\underset{\sim}{2}_{\mathcal{T}}$ is injective in the category $\mathfrak{X}_{01}$.

The ordered set $\mathbf{D}(L,\underline{2})$ is dually order-isomorphic to $\mathcal{I}_{\mathrm{p}}(L) \mathbin{\dot\cup} \{0, 1\}$, via the map $x \mapsto x^{-1}(0)$, and, for every non-trivial bounded ordered set P, the lattice $\mathcal{P}_{01}(P,\underset{\sim}{2})$ is dually order-isomorphic to $\mathcal{O}(P\backslash\{0, 1\})$, via the map $\alpha \mapsto \alpha^{-1}(0)$. That the map $e : L \to \mathcal{O}(\mathcal{I}_{\mathrm{p}}(L))$ is an embedding is now an easy consequence of the fact that the map e from (a) is an embedding. □

Thus, the formation of canonical extensions of distributive lattices may be viewed as a hybrid between the Priestley and Banaschewski dualities. The following property of canonical extensions is an immediate corollary. Let $^\flat : \mathfrak{X}_{01} \to \mathcal{P}_{01}$ be the forgetful functor that maps every bounded Priestley space to its underlying ordered set.

Corollary 4.5 (Gehrke and Jónsson [25, 3.2]) *The formation of canonical extensions is functorial from the category $\mathbf{D}$ of distributive lattices to the category $\mathbf{L}$ of Boolean topological distributive lattices, or equivalently, to the category $\mathbf{C}$ of doubly algebraic distributive lattices.*

Proof By (a) of Theorem 4.4, the lattice $E(H(L)^\flat)$ is a canonical extension of L. Since H, $^\flat$ and E are functors, the result follows. □

We close this section by showing how our approach leads to a conceptually simple proof of the fact, proved for bounded distributive lattices in Bezhanishvili et al. [3], that the profinite completion and the canonical extension of a distributive lattice coincide.

Given a distributive lattice L, let $\mathcal{S}_L$ be the set of all congruences on L of finite index. For $\theta_1, \theta_2 \in \mathcal{S}_L$ with $\theta_1 \subseteq \theta_2$, let $\varphi_{\theta_1\theta_2} : L/\theta_1 \to L/\theta_2$ be the natural map. Thus, by ordering the set $\mathcal{S}_L$ by reverse inclusion and taking these homomorphisms as the connecting maps, the set of all factor algebras L/θ, for $\theta \in \mathcal{S}_L$, forms an inverse system in $\mathbf{D}$. The *profinite completion* of L is defined to be the inverse limit of this inverse system.

The proof below requires the following well-known description of congruences on a distributive lattice via Priestley duality. Let L be a distributive lattice. Applying Priestley duality, we may assume that $L = K(X)$, for some nontrivial bounded Priestley space X. A subset Y of a bounded ordered set that includes the bounds will be called a 0,1-*subset*. For every closed 0,1-subset Y of X, define a congruence θ_Y on $K(X)$ by

$$\alpha \equiv \beta\ (\theta_Y) \iff \alpha{\restriction}_Y = \beta{\restriction}_Y.$$

Then the map $Y \mapsto \theta_Y$ is a dual lattice isomorphism between the lattice of closed 0, 1-subsets on X and the lattice of congruences on $L = K(X)$: more generally, see Clark and Davey [8, 3.2.1]. Moreover, we have $H(L/\theta_Y) \cong Y$. Thus the set of congruences of finite index on $L = K(X)$ is given by $\mathcal{S}_L = \{\theta_Y \mid Y \Subset X\}$.

Theorem 4.6 (Bezhanishvili et al. [3]) *Let L be a distributive lattice. Then the profinite completion of L is a canonical extension of L.*

Proof Without loss of generality, let $L = K(X)$, for some bounded Priestley space X (recall that $K = \mathfrak{X}_{01}(-, \underset{\sim}{2}_{\mathcal{T}}) : \mathfrak{X}_{01} \to \mathfrak{L}$). The calculations below use two easy facts. First, that every bounded ordered set P is the direct limit in $\mathfrak{P}_{01}$ of its finite 0, 1-subsets (with the order on each subset induced from the order on P), and second, that if M is a finite distributive lattice, then the Priestley and Banaschewski duals of M differ only by the removal of the discrete topology: in symbols, $D(M) = H(M)^\flat$. All of the direct limits in the next line are calculated in $\mathfrak{P}_{01}$.

$$X^\flat \cong \varinjlim_{Y \Subset X} Y \cong \varinjlim_{Y \Subset X} H(L/\theta_Y)^\flat = \varinjlim_{Y \Subset X} D(L/\theta_Y) \cong \varinjlim_{\theta \in \mathcal{S}_L} D(L/\theta).$$

Since we now apply the functor E, the inverse limits in the next line are formed in $\mathfrak{L}$.

$$E(X^\flat) \cong E\left(\varinjlim_{\theta \in \mathcal{S}_L} D(L/\theta)\right) \cong \varprojlim_{\theta \in \mathcal{S}_L} ED(L/\theta) \cong \varprojlim_{\theta \in \mathcal{S}_L} L/\theta. \qquad (\dagger)$$

As L/θ is finite, for all $\theta \in \mathcal{S}_L$, the underlying lattice of the Boolean topological lattice $\varprojlim^{\mathfrak{L}}_{\theta \in \mathcal{S}_L} L/\theta$ is isomorphic to the lattice $\varprojlim^{\mathfrak{D}}_{\theta \in \mathcal{S}_L} L/\theta$. Hence, by Eq. †, the lattice $E(X^\flat)$ is isomorphic to the profinite completion of L. Since, by Theorem 4.4, $E(X^\flat) \cong E(HK(X)^\flat) = E(H(L)^\flat)$ is a canonical extension of L, the result follows. □

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