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Using coloured ordered sets to study finite-level full dualities

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Abstract. We consider all the full dualities for the class of finite bounded distributive lattices that are based on the three-element chain $\underline{\mathbf{3}}$. Under a natural quasi-order, these full dualities form a doubly algebraic lattice $\mathcal{F}_{\underline{\mathbf{3}}}$. Using Priestley duality, we establish a correspondence between the elements of $\mathcal{F}_{\underline{\mathbf{3}}}$ and special enriched ordered sets, which we call 'coloured ordered sets'. We can then use combinatorial arguments to show that the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ has cardinality $2^{\aleph_0}$ and is non-modular. This is the first investigation into the structure of an infinite lattice of finite-level full dualities.

1. Introduction

How much choice did H. A. Priestley have in setting up her duality [16, 17] for the variety $\mathfrak{D}$ of bounded distributive lattices? There is a concrete sense in which there is only one full duality for $\mathfrak{D}$ based on the two-element bounded lattice $\underline{\mathbf{2}}$ (see Example 2.5). We shall show that, in contrast, there are uncountably many different full dualities for the class $\mathfrak{D}_{\text{fin}}$ of *finite* bounded distributive lattices based on the *three-element* bounded lattice $\underline{\mathbf{3}}$.

The lattice $\underline{\mathbf{3}}$ has been a seminal example throughout the development of natural duality theory [3, 4, 7]. Recently, in work on the Full Versus Strong Problem, the lattice $\underline{\mathbf{3}}$ was the basis for the first example of a finite-level duality that is full but not strong (Davey, Haviar and Willard [10]).

In general, a finite algebra $\underline{\mathbf{M}}$ admits essentially only one finite-level strong duality, but can admit many different finite-level full dualities. These finite-level full dualities form a doubly algebraic lattice $\mathcal{F}_{\underline{\mathbf{M}}}$. This lattice, introduced by Davey, Pitkethly and Willard [12], is constructed by imposing a very natural quasi-order on the collection of all alter egos that fully dualise $\underline{\mathbf{M}}$ at the finite level.

In this paper, we study the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of all finite-level full dualities based on $\underline{\mathbf{3}}$. We show that this lattice has size $2^{\aleph_0}$ by order-embedding the powerset $\wp(\mathbb{N})$ into $\mathcal{F}_{\underline{\mathbf{3}}}$ (Section 6). We also show that $\mathcal{F}_{\underline{\mathbf{3}}}$ is non-modular (Section 9). The complexity of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is somewhat surprising, given that the corresponding lattice for the two-element bounded lattice, $\mathcal{F}_{\underline{\mathbf{2}}}$, has size 1.

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Indeed, this is the first investigation into the structure of an *infinite* lattice of finite-level full dualities. The lattice $\mathcal{F}_{\underline{\mathbf{M}}}$ is known to have size 1 whenever $\underline{\mathbf{M}}$ is a finite semilattice or abelian group (Davey, Haviar and Niven [6]), or a finite algebra injective in the quasivariety it generates (Davey and Werner [15]). The lattice $\mathcal{F}_{\underline{\mathbf{M}}}$ is finite for any quasi-primal algebra $\underline{\mathbf{M}}$ [12].

We study the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ indirectly, using Priestley duality. In Section 5, we establish a correspondence between the alter egos that fully dualise $\underline{\mathbf{3}}$ at the finite level and certain enriched ordered sets, which we call 'coloured ordered sets'. These are structures of the form $\langle C; \leqslant, \lhd \rangle$, where $\leqslant$ is an order on C and $\lhd$ is a subset of $\leqslant$ satisfying certain conditions (Definition 5.1). A finite coloured ordered set $\langle C; \leqslant, \lhd \rangle$ can be presented pictorially, by starting with the ordered-set diagram of $\langle C; \leqslant \rangle$ and adding a 'coloured edge' between the elements of each pair in $\lhd$. (See the diagrams in Section 5 onwards.)

We shall set up a quasi-order on coloured ordered sets that corresponds exactly to the natural quasi-order on alter egos (Definition 5.6). Having done this, we can study the structure of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ via a purely combinatorial analysis of coloured ordered sets.

2. General background

We shall translate our duality-theoretic problem into a combinatorial problem. The translation process, developed over Sections 4 and 5, requires some familiarity with the finite-level version of Priestley duality. We begin with a brief refresher, and refer to Clark–Davey [1] for more details.

Priestley duality 2.1. Priestley duality provides a natural correspondence between $\mathfrak{D}_{\text{fin}}$, the class of all finite bounded distributive lattices, and $\mathfrak{P}_{\text{fin}}$, the class of all finite ordered sets. This correspondence is based on the two-element bounded lattice $\underline{\mathbf{2}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$ and the two-element ordered set $\underset{\sim}{\mathbf{2}} = \langle \{0, 1\}; \leqslant \rangle$, where $0 < 1$. We use

$$\mathrm{H}\colon \mathfrak{D}_{\text{fin}} \to \mathfrak{P}_{\text{fin}} \quad \text{and} \quad \mathrm{K}\colon \mathfrak{P}_{\text{fin}} \to \mathfrak{D}_{\text{fin}}$$

for the associated hom-functors:

- the dual of a finite lattice $\mathbf{L}$ is $\mathrm{H}(\mathbf{L}) = \langle \hom(\mathbf{L}, \underline{\mathbf{2}}); \leqslant \rangle$, where the order is defined pointwise from $\underset{\sim}{\mathbf{2}}$;
- the dual of a finite ordered set $\mathbf{P}$ is $\mathrm{K}(\mathbf{P}) = \langle \hom(\mathbf{P}, \underset{\sim}{\mathbf{2}}); \vee, \wedge, 0, 1 \rangle$, where the lattice operations are defined pointwise from $\underline{\mathbf{2}}$;
- the dual of a lattice homomorphism $\varphi\colon \mathbf{L}_1 \to \mathbf{L}_2$ is the order-preserving map $\mathrm{H}(\varphi)\colon \mathrm{H}(\mathbf{L}_2) \to \mathrm{H}(\mathbf{L}_1)$ given by $\mathrm{H}(\varphi)(x) = x \circ \varphi$; the dual of an order-preserving map is defined similarly.

This pair of hom-functors provides a dual equivalence between $\mathfrak{D}_{\text{fin}}$ and $\mathfrak{P}_{\text{fin}}$. In particular, there is a natural isomorphism $e_{\mathbf{L}}\colon \mathbf{L} \to \mathrm{KH}(\mathbf{L})$, for each $\mathbf{L}$ in $\mathfrak{D}_{\text{fin}}$, defined by $e_{\mathbf{L}}(a)(x) = x(a)$. Similarly, there is a natural isomorphism $\varepsilon_{\mathbf{P}}\colon \mathbf{P} \to \mathrm{HK}(\mathbf{P})$, for each $\mathbf{P}$ in $\mathfrak{P}_{\text{fin}}$.

Priestley duality, as described above, is a particular example of a natural duality. We next provide the minimum amount of background in natural dualities that is needed to set up our problem. For a detailed discussion of this particular background material, including the proofs, see the recent paper by Davey, Pitkethly and Willard [12]. For a comprehensive treatment of natural dualities in general, including motivations and deep applications, see the Clark–Davey monograph [1].

Alter egos 2.2. Fix a finite algebra $\underline{\mathbf{M}} = \langle M; F\rangle$.

- For all $n \geqslant 1$, an n-ary relation r on M is an *algebraic relation* on $\underline{\mathbf{M}}$ if it is the underlying set of a subalgebra $\mathbf{r}$ of $\underline{\mathbf{M}}^n$.
- For all $n \geqslant 0$, an n-ary partial operation $h\colon r \to M$ on M is an *algebraic partial operation* on $\underline{\mathbf{M}}$ if the domain r is the underlying set of a subalgebra $\mathbf{r}$ of $\underline{\mathbf{M}}^n$ and $h\colon \mathbf{r} \to \underline{\mathbf{M}}$ is a homomorphism.

An *alter ego* of $\underline{\mathbf{M}}$ is a partial algebra $\underset{\sim}{\mathbf{M}} = \langle M; H\rangle$ such that H is a set of algebraic partial operations on $\underline{\mathbf{M}}$. We can also include algebraic relations in the type of an alter ego: they simply correspond to algebraic partial operations that are projections. (In general, an alter ego is endowed with the discrete topology. But topology will play no role in this paper, as we only work at the finite level.)

As in Priestley duality, the choice of an alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$ induces a pair of hom-functors $\mathrm{D}\colon \boldsymbol{\mathcal{A}}_{\mathrm{fin}} \to \boldsymbol{\mathcal{X}}_{\mathrm{fin}}$ and $\mathrm{E}\colon \boldsymbol{\mathcal{X}}_{\mathrm{fin}} \to \boldsymbol{\mathcal{A}}_{\mathrm{fin}}$, where

- $\boldsymbol{\mathcal{A}}_{\mathrm{fin}}$ consists of all algebras that embed into $\underline{\mathbf{M}}^n$, for some $n \geqslant 0$, and
- $\boldsymbol{\mathcal{X}}_{\mathrm{fin}}$ consists of all (possibly empty) partial algebras that embed into $\underset{\sim}{\mathbf{M}}^n$, for some $n \geqslant 1$.

If, as in Priestley duality, the natural embeddings $e_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{ED}(\mathbf{A})$ and $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to \mathrm{DE}(\mathbf{X})$ are isomorphisms, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}_{\mathrm{fin}}$ and $\mathbf{X} \in \boldsymbol{\mathcal{X}}_{\mathrm{fin}}$, then we say that the alter ego $\underset{\sim}{\mathbf{M}}$ *fully dualises* $\underline{\mathbf{M}}$ *at the finite level.*

Quasi-order on alter egos 2.3. To define the *term-reduct* quasi-order on the collection of all algebras on the set M, we only need to say what we mean by the *term functions* of an algebra. Analogously, to define the *structural-reduct* quasi-order on the collection of all alter egos of the algebra $\underline{\mathbf{M}}$, we only need to say what we mean by the *structural functions* of an alter ego.

Let $\underset{\sim}{\mathbf{M}} = \langle M; H\rangle$ be an alter ego of $\underline{\mathbf{M}}$. The *partial clone* of $\underset{\sim}{\mathbf{M}}$ is the smallest set of partial operations on M that contains H, contains all total projection operations on M and is closed under composition (with maximum domain).

Let $h\colon r \to M$ be an n-ary algebraic partial operation on $\underline{\mathbf{M}}$, for some $n \geqslant 0$. We say that h is a *structural function* of the alter ego $\underset{\sim}{\mathbf{M}}$ if

- h has an extension in the partial clone of $\underset{\sim}{\mathbf{M}}$, and
- r is conjunct-atomic definable from $\underset{\sim}{\mathbf{M}}$ (that is, the subset r of M^n can be described in $\underset{\sim}{\mathbf{M}}$ by a conjunction of atomic formulæ of type H).

In particular, each $h \in H$ is a structural function of $\underset{\sim}{\mathbf{M}} = \langle M; H\rangle$.

We now say that an alter ego $\underset{\sim}{\mathbf{M}}_1$ is a *structural reduct* of an alter ego $\underset{\sim}{\mathbf{M}}_2$, and we write $\underset{\sim}{\mathbf{M}}_1 \sqsubseteq \underset{\sim}{\mathbf{M}}_2$, if every structural function of $\underset{\sim}{\mathbf{M}}_1$ is also a structural function of $\underset{\sim}{\mathbf{M}}_2$. To show that $\underset{\sim}{\mathbf{M}}_1 \sqsubseteq \underset{\sim}{\mathbf{M}}_2$, where $\underset{\sim}{\mathbf{M}}_1 = \langle M; H_1\rangle$, it is enough to show that every partial operation in H_1 is a structural function of $\underset{\sim}{\mathbf{M}}_2$.

This quasi-order induces a natural equivalence on the alter egos of $\underline{\mathbf{M}}$. We say that alter egos $\underset{\sim}{\mathbf{M}}_1$ and $\underset{\sim}{\mathbf{M}}_2$ are *structurally equivalent*, and we write $\underset{\sim}{\mathbf{M}}_1 \equiv \underset{\sim}{\mathbf{M}}_2$, if they have exactly the same structural functions. We shall write $\underset{\sim}{\mathbf{M}}_1 \sqsubset \underset{\sim}{\mathbf{M}}_2$ to mean that $\underset{\sim}{\mathbf{M}}_1 \sqsubseteq \underset{\sim}{\mathbf{M}}_2$ and $\underset{\sim}{\mathbf{M}}_1 \not\equiv \underset{\sim}{\mathbf{M}}_2$.

(Note that our definition of structural reduct is equivalent to that used by Davey, Haviar and Willard [11], and therefore our definition of structural equivalence is equivalent to that first introduced by Clark and Davey [1].)

Lattice of alter egos 2.4. The set of all alter egos of $\underline{\mathbf{M}}$ with the structural-reduct quasi-order can be factored, in the natural way, to obtain an ordered set $\mathcal{A}_{\underline{\mathbf{M}}}$. In fact, the ordered set $\mathcal{A}_{\underline{\mathbf{M}}}$ of (equivalence classes of) alter egos forms a doubly algebraic lattice [12, §2]. The join in the lattice $\mathcal{A}_{\underline{\mathbf{M}}}$ corresponds to the union of representatives [12, 2.7]. More precisely, for alter egos $\underset{\sim}{\mathbf{M}}_i = \langle M; H_i\rangle$ of $\underline{\mathbf{M}}$, indexed by a non-empty set I, we have

$$\bigvee_{i\in I} (\underset{\sim}{\mathbf{M}}_i/{\equiv}) = \big(\bigcup_{i\in I} \underset{\sim}{\mathbf{M}}_i\big)/{\equiv}, \quad \text{where } \bigcup_{i\in I} \underset{\sim}{\mathbf{M}}_i := \big\langle M; \bigcup_{i\in I} H_i\big\rangle.$$

The meet in $\mathcal{A}_{\underline{\mathbf{M}}}$ cannot be calculated easily via representatives [12, 2.7].

Our equivalence on the alter egos of $\underline{\mathbf{M}}$ is compatible with the basic concepts of duality theory. In particular, if one alter ego in an equivalence class yields a finite-level full duality, then every other alter ego in the equivalence class does too. So we can consider the subordered set $\mathcal{F}_{\underline{\mathbf{M}}}$ of the lattice $\mathcal{A}_{\underline{\mathbf{M}}}$ consisting of the equivalence classes of alter egos that fully dualise $\underline{\mathbf{M}}$ at the finite level. In fact, the ordered set $\mathcal{F}_{\underline{\mathbf{M}}}$ forms a complete sublattice of $\mathcal{A}_{\underline{\mathbf{M}}}$ [12, 5.5].

The top element of $\mathcal{F}_{\underline{\mathbf{M}}}$ is represented by the alter ego $\underset{\sim}{\mathbf{M}}_\Omega := \langle M; H_\Omega\rangle$, where H_Ω is the set of all algebraic partial operations on $\underline{\mathbf{M}}$. Up to structural equivalence, this is the only alter ego that strongly dualises $\underline{\mathbf{M}}$ at the finite level [12, 4.6].

Example 2.5. [12, 3.5] Up to structural equivalence, the bounded lattice $\underline{\mathbf{2}}$ has only two alter egos: the chain $\underset{\sim}{\mathbf{2}} = \langle\{0,1\}; \leqslant\rangle$ and the set $\underset{\sim}{\mathbf{2}}_\varnothing = \langle\{0,1\};\,\rangle$. The alter ego $\underset{\sim}{\mathbf{2}}$ fully dualises $\underline{\mathbf{2}}$ at the finite level, but $\underset{\sim}{\mathbf{2}}_\varnothing$ does not. So the lattice $\mathcal{F}_{\underline{\mathbf{2}}}$ has size 1.

Our main aim in this paper is to prove that, in contrast with the example above, the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is uncountable.

We rely on two general results about finite-level full dualities. The first, which helps us to recognise structural functions, follows from results due to Davey, Haviar and Willard [11, 3.3 and 3.9]. Note that, for an n-ary algebraic relation r on $\underline{\mathbf{M}}$, the set $\hom(\mathbf{r}, \underline{\mathbf{M}})$ consists of all algebraic partial operations on $\underline{\mathbf{M}}$ with domain r and includes the projections $\rho_1, \dots, \rho_n \colon r \to M$.

Useful Lemma 2.6. *Let $\underline{\mathbf{M}}$ be a finite algebra. Assume that $\underset{\sim}{\mathbf{M}}$ fully dualises $\underline{\mathbf{M}}$ at the finite level. For each algebraic relation r on $\underline{\mathbf{M}}$, the following are equivalent*:

(1) *every partial operation in* $\hom(\mathbf{r}, \underline{\mathbf{M}})$ *is a structural function of* $\underset{\sim}{\mathbf{M}}$;
(2) *some partial operation in* $\hom(\mathbf{r}, \underline{\mathbf{M}})$ *is a structural function of* $\underset{\sim}{\mathbf{M}}$;
(3) *the relation r is conjunct-atomic definable from* $\underset{\sim}{\mathbf{M}}$;
(4) *the set* $\hom(\mathbf{r}, \underline{\mathbf{M}})$ *is generated by the projections in the direct power* $\underset{\sim}{\mathbf{M}}^r$.

The second result, below, gives us access to all the elements of the lattice $\mathcal{F}_{\underline{\mathbf{M}}}$, provided we have knowledge about the bottom element.

Full Enrichment Theorem 2.7. [12, 5.3] *Let $\underline{\mathbf{M}}$ be a finite algebra. Assume that $\underset{\sim}{\mathbf{M}}_1 = \langle M; H_1\rangle$ fully dualises $\underline{\mathbf{M}}$ at the finite level, and let $\underset{\sim}{\mathbf{M}}_2 = \langle M; H_2\rangle$ be an alter ego of $\underline{\mathbf{M}}$ with $\underset{\sim}{\mathbf{M}}_1 \sqsubseteq \underset{\sim}{\mathbf{M}}_2$. The following are equivalent*:

(1) $\underset{\sim}{\mathbf{M}}_2$ *fully dualises $\underline{\mathbf{M}}$ at the finite level*;
(2) *for all $h \in H_2 \backslash H_1$, every algebraic partial operation on $\underline{\mathbf{M}}$ with the same domain as h is a structural function of $\underset{\sim}{\mathbf{M}}_2$*;
(3) *for each relation r that is the domain of a partial operation in $H_2 \backslash H_1$, the set* $\hom(\mathbf{r}, \underline{\mathbf{M}})$ *is generated by the projections in* $(\underset{\sim}{\mathbf{M}}_2)^r$.

Note that the implication (1) ⇒ (2) of this theorem comes straight from the Useful Lemma. The reverse implication is the surprising part, and gives us the following corollary.

Corollary 2.8. *Let $\underline{\mathbf{M}}$ be a finite algebra, and assume that $\underset{\sim}{\mathbf{M}}_1 = \langle M; H_1\rangle$ fully dualises $\underline{\mathbf{M}}$ at the finite level. Let R be a set of algebraic relations on $\underline{\mathbf{M}}$, and define $H_2 := H_1 \cup \bigcup_{r\in R} \hom(\mathbf{r}, \underline{\mathbf{M}})$. Then $\underset{\sim}{\mathbf{M}}_2 := \langle M; H_2\rangle$ fully dualises $\underline{\mathbf{M}}$ at the finite level.*

3. A first look at the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$

We now start to look at the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities. (As in the last section, we view $\mathcal{F}_{\underline{\mathbf{3}}}$ as a lattice of alter egos of $\underline{\mathbf{3}}$, modulo the natural equivalence.) We begin by summarising what is already known about $\mathcal{F}_{\underline{\mathbf{3}}}$.

Define the three-element bounded lattice

$$\underline{\mathbf{3}} := \langle \{0, d, 1\}; \vee, \wedge, 0, 1\rangle,$$

where $0 < d < 1$. Define the two endomorphisms f and g of $\underline{\mathbf{3}}$ and the binary algebraic partial operations h and σ on $\underline{\mathbf{3}}$ as in Figure 1. These four partial operations play a very important role throughout the paper.

The alter ego $\underset{\sim}{\mathbf{3}}_0 := \langle \{0, d, 1\}; f, g\rangle$ of $\underline{\mathbf{3}}$ yields a duality on the variety $\boldsymbol{\mathfrak{D}}$ of bounded distributive lattices, but does not even yield a full duality on $\boldsymbol{\mathfrak{D}}_{\text{fin}}$ (Davey, Haviar and Priestley [8]).

Bottom of $\mathcal{F}_{\underline{\mathbf{3}}}$ 3.1. The alter ego $\underset{\sim}{\mathbf{3}}_h := \langle \{0, d, 1\}; f, g, h\rangle$ yields a full duality on $\boldsymbol{\mathfrak{D}}_{\text{fin}}$ (Davey, Haviar and Willard [10]). In fact, every alter ego that fully

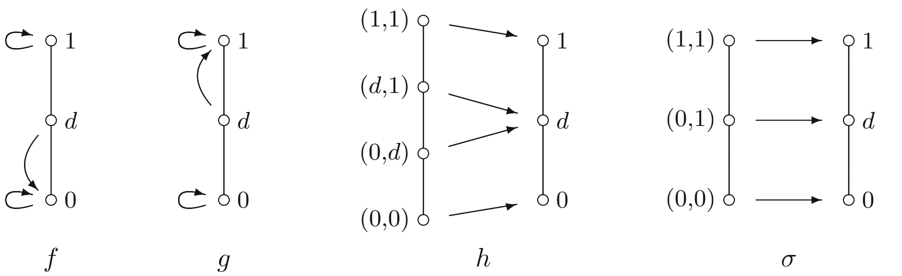

FIGURE 1. The algebraic partial operations f, g, h and σ on $\underline{\mathbf{3}}$

dualises $\underline{\mathbf{3}}$ at the finite level must have f, g and h as structural functions. To see why, assume that $\underset{\sim}{\mathbf{3}}$ fully dualises $\underline{\mathbf{3}}$ at the finite level. The unary relation $\{0, d, 1\}$ is trivially conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}$. So the unary algebraic operations f and g are structural functions of $\underset{\sim}{\mathbf{3}}$, by the Useful Lemma, 2.6. Using f and g, we can describe the domain of h:

$$\operatorname{dom}(h) = \{\, (x, y) \in \{0, d, 1\}^2 \mid g(x) = f(y) \,\}.$$

Now, using the Useful Lemma again, we obtain h as a structural function of $\underset{\sim}{\mathbf{3}}$. It follows that $\underset{\sim}{\mathbf{3}}_h$ is a representative for the bottom element of $\mathcal{F}_{\underline{\mathbf{3}}}$.

It is important to note that, whereas every enrichment of $\underset{\sim}{\mathbf{3}}_0$ yields a duality on $\mathfrak{D}$, not every enrichment of $\underset{\sim}{\mathbf{3}}_h$ yields a full duality on $\mathfrak{D}_{\text{fin}}$. (We illustrate this in Example 4.12.) In other words, the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ does not form an increasing subset of $\mathcal{A}_{\underline{\mathbf{3}}}$. Nevertheless, the alter ego $\underset{\sim}{\mathbf{3}}_h$ is the jumping-off point for our study of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$. Via the Full Enrichment Theorem, 2.7, we can use $\underset{\sim}{\mathbf{3}}_h$ to access all elements of $\mathcal{F}_{\underline{\mathbf{3}}}$.

Top of $\mathcal{F}_{\underline{\mathbf{3}}}$ 3.2. The alter ego $\underset{\sim}{\mathbf{3}}_\sigma := \langle \{0, d, 1\}; f, g, \sigma \rangle$ yields a strong duality on $\mathfrak{D}$ (Davey and Haviar [4]). So $\underset{\sim}{\mathbf{3}}_\sigma$ is a representative for the top element of $\mathcal{F}_{\underline{\mathbf{3}}}$. In other words, the alter ego $\underset{\sim}{\mathbf{3}}_\sigma$ is structurally equivalent to the top alter ego $\underset{\sim}{\mathbf{3}}_\Omega$.

In the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities, the top element is the only one that actually gives a full duality for the whole variety $\mathfrak{D}$ of bounded distributive lattices (Davey, Haviar and Willard [10]).

Coatom of $\mathcal{F}_{\underline{\mathbf{3}}}$ 3.3. We shall say that an n-ary algebraic relation r on $\underline{\mathbf{3}}$ is *quasi-boolean* if each element of $r \cap \{0, 1\}^n$ has a complement in the sublattice $\mathbf{r}$ of $\underline{\mathbf{3}}^n$. Now let H_β denote the set of all algebraic partial operations on $\underline{\mathbf{3}}$ with quasi-boolean domain, and define the alter ego $\underset{\sim}{\mathbf{3}}_\beta := \langle \{0, d, 1\}; H_\beta \rangle$. We finish this section by showing that $\underset{\sim}{\mathbf{3}}_\beta$ is the unique coatom of $\mathcal{F}_{\underline{\mathbf{3}}}$.

(For any finite non-boolean bounded distributive lattice $\underline{\mathbf{L}}$, an analogous alter ego $\underset{\sim}{\mathbf{L}}_\beta$ yields a finite-level full but not strong duality [12, 5.10].)

Lemma 3.4. *The top element of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is completely join-irreducible, and the alter ego $\underset{\sim}{\mathbf{3}}_\beta$ is a representative for its unique lower cover.*

Proof. First note that $f, g, h \in H_\beta$. Since $\underset{\sim}{\mathbf{3}}_h$ fully dualises $\underline{\mathbf{3}}$ at the finite level, it follows from Corollary 2.8 that $\underset{\sim}{\mathbf{3}}_\beta$ also fully dualises $\underline{\mathbf{3}}$ at the finite level. So $\underset{\sim}{\mathbf{3}}_\beta$ represents an element of $\mathcal{F}_{\underline{\mathbf{3}}}$.

Next we check that $\underset{\sim}{\mathbf{3}}_\beta$ is not structurally equivalent to the top alter ego $\underset{\sim}{\mathbf{3}}_\Omega$. The set $X := \{0,1\}$ does not form a substructure of $\underset{\sim}{\mathbf{3}}_\Omega$, since X is not closed under the partial operation σ. We shall show that X forms a substructure of $\underset{\sim}{\mathbf{3}}_\beta$. Since substructures are closed under structural functions, it will then follow that σ is not a structural function of $\underset{\sim}{\mathbf{3}}_\beta$, and thus that $\underset{\sim}{\mathbf{3}}_\beta \not\equiv \underset{\sim}{\mathbf{3}}_\Omega$.

Consider an n-ary partial operation $k\colon \mathbf{r} \to \underline{\mathbf{3}}$ in H_β and let $a \in r \cap X^n = r \cap \{0,1\}^n$. As r is quasi-boolean, there is a complement c for a in the lattice $\mathbf{r}$. As k is a homomorphism, the elements $k(a)$ and $k(c)$ are complements in $\underline{\mathbf{3}}$. Thus $k(a) \in \{0,1\} = X$. So X forms a substructure of $\underset{\sim}{\mathbf{3}}_\beta$, whence $\underset{\sim}{\mathbf{3}}_\beta \not\equiv \underset{\sim}{\mathbf{3}}_\Omega$.

Now let $\underset{\sim}{\mathbf{3}}$ be an alter ego that fully dualises $\underline{\mathbf{3}}$ at the finite level. We can finish the proof by showing that

(∗) if $\underset{\sim}{\mathbf{3}} \not\equiv \underset{\sim}{\mathbf{3}}_\Omega$, then $\underset{\sim}{\mathbf{3}}$ is a reduct of $\underset{\sim}{\mathbf{3}}_\beta$ (and therefore $\underset{\sim}{\mathbf{3}} \sqsubseteq \underset{\sim}{\mathbf{3}}_\beta$).

We prove the contrapositive. Assume that there is an n-ary partial operation $k\colon \mathbf{r} \to \underline{\mathbf{3}}$ in the type of $\underset{\sim}{\mathbf{3}}$, for some $n \geqslant 1$, such that r is not quasi-boolean. We want to show that $\underset{\sim}{\mathbf{3}} \equiv \underset{\sim}{\mathbf{3}}_\Omega$. Since $\underset{\sim}{\mathbf{3}}_\Omega \equiv \underset{\sim}{\mathbf{3}}_\sigma$, it is enough to show that f, g and σ are structural functions of $\underset{\sim}{\mathbf{3}}$.

As $\underset{\sim}{\mathbf{3}}$ fully dualises $\underline{\mathbf{3}}$ at the finite level, we know that it has f and g as structural functions (see 3.1). By the Useful Lemma, 2.6, it now remains to prove that the domain of σ is conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}$.

As r is not quasi-boolean, there is an element $a = (a_1, \dots, a_n)$ of $r \cap \{0,1\}^n$ that does not have a complement in the lattice $\mathbf{r}$. So $a' \notin r$, where a' denotes the pointwise complement of a in $\{0,1\}^n$. We claim that

$$\operatorname{dom}(\sigma) = \bigl\{\, (x_0, x_1) \in \{0,1\}^2 \bigm| (x_{a_1}, x_{a_2}, \dots, x_{a_n}) \in r \,\bigr\}.$$

(To see this, note that $(x_0, x_1) = (0,1)$ implies $(x_{a_1}, x_{a_2}, \dots, x_{a_n}) = a \in r$, and that $(x_0, x_1) = (1,0)$ implies $(x_{a_1}, x_{a_2}, \dots, x_{a_n}) = a' \notin r$.) As the unary relation $\{0,1\}$ on $\underline{\mathbf{3}}$ is described by the atomic formula $f(x) = x$, it now follows that $\operatorname{dom}(\sigma)$ is conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}$, as required. □

4. Studying $\mathcal{F}_{\underline{\mathbf{3}}}$ via labelled ordered sets

In this section, we use Priestley duality to encode algebraic relations and partial operations on $\underline{\mathbf{3}}$ as labelled ordered sets. This gives us an initial tool for studying the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$, and we illustrate its use by showing that $\mathcal{F}_{\underline{\mathbf{3}}}$ contains an infinite ascending chain. We shall refine this tool in the next section.

Priestley duality has been used to encode algebraic relations on Kleene algebras [9], Heyting algebras [14] and distributive p-algebras [5, 13].

Definition 4.1. Consider an ordered set $\mathbf{P} = \langle P; \leqslant \rangle$. We define a 2-*chain* in $\mathbf{P}$ to be a subset $\{a, b\}$ of P, where $a \leqslant b$. (We allow the case $a = b$.) We say that a collection $\bigl\{ \{a_i, b_i\} \bigm| i \in I \bigr\}$ of 2-chains *covers* $\mathbf{P}$ if $P = \bigcup_{i \in I} \{a_i, b_i\}$.

Given $\mathbf{L} \in \mathfrak{D}_{\text{fin}}$ and a homomorphism $x\colon \mathbf{L} \to \underline{\mathbf{3}}$, we can define the 2-chain $\widehat{x} := \{f \circ x, g \circ x\}$ in the Priestley dual $\mathrm{H}(\mathbf{L})$.

Lemma 4.2. *Let $\mathbf{L} \in \mathfrak{D}_{\text{fin}}$. There is a bijection from the set* $\hom(\mathbf{L}, \underline{\mathbf{3}})$ *to the set of all* 2*-chains of* $\mathrm{H}(\mathbf{L})$, *given by $x \mapsto \widehat{x}$.*

Sketch proof. Priestley duality gives us a bijection between the sets $\hom(\mathbf{L}, \underline{\mathbf{3}})$ and $\hom(\mathrm{H}(\underline{\mathbf{3}}), \mathrm{H}(\mathbf{L}))$. The dual $\mathrm{H}(\underline{\mathbf{3}})$ is the two-element chain $\langle \{f, g\}; \leqslant \rangle$, where $f < g$. So we can identify a homomorphism $x\colon \mathbf{L} \to \underline{\mathbf{3}}$ with the image of its dual $\mathrm{H}(x)\colon \mathrm{H}(\underline{\mathbf{3}}) \to \mathrm{H}(\mathbf{L})$, which is a 2-chain in $\mathrm{H}(\mathbf{L})$. As $\mathrm{H}(x)(f) = f \circ x$ and $\mathrm{H}(x)(g) = g \circ x$, we are identifying x with $\widehat{x}$. ☐

Lemma 4.3. *Let $\mathbf{L} \in \mathfrak{D}_{\text{fin}}$ and let $x_1, \dots, x_n\colon \mathbf{L} \to \underline{\mathbf{3}}$, for some $n \geqslant 1$. Then the following are equivalent*:

(1) *the homomorphisms $x_1, \dots, x_n$ separate the elements of* $\mathbf{L}$;
(2) *the* 2*-chains $\widehat{x}_1, \dots, \widehat{x}_n$ cover* $\mathrm{H}(\mathbf{L})$.

Sketch proof. Assume (2), and let $a \neq b$ in L. There exists $y\colon \mathbf{L} \to \underline{\mathbf{2}}$ with $y(a) \neq y(b)$. By (2), we have $y \in \widehat{x}_i = \{f \circ x_i, g \circ x_i\}$, for some $i \in \{1, \dots, n\}$. So $x_i(a) \neq x_i(b)$. Thus (1) holds.

Now assume (1). The natural product map $\xi := x_1 \sqcap \dots \sqcap x_n\colon \mathbf{L} \to \underline{\mathbf{3}}^n$ is an embedding. Choose any $y\colon \mathbf{L} \to \underline{\mathbf{2}}$ in $\mathrm{H}(\mathbf{L})$. As $\underline{\mathbf{2}}$ is injective in $\mathfrak{D}_{\text{fin}}$, there exists $z\colon \underline{\mathbf{3}}^n \to \underline{\mathbf{2}}$ with $y = z \circ \xi$. The algebraic operation z must be essentially unary. (To check this, use the correspondence under Priestley duality between products in $\mathfrak{D}_{\text{fin}}$ and disjoint unions in $\mathfrak{P}_{\text{fin}}$.) Say that z only depends on the ith coordinate. It follows that $y \in \{f \circ x_i, g \circ x_i\} = \widehat{x}_i$. Thus (2) holds. ☐

We can now explain how to encode an algebraic relation r on $\underline{\mathbf{3}}$ as a labelled ordered set. First, note that the abstract ordered set $\mathrm{H}(\mathbf{r})$ only captures the lattice $\mathbf{r}$ up to isomorphism, it does not capture the concrete relation r. For example, the two different relations $\mathrm{dom}(h)$ and $\mathrm{graph}(h)$ have isomorphic ordered sets as their Priestley duals.

Correspondence 4.4. *Let $n \geqslant 1$. The n-ary algebraic relations on $\underline{\mathbf{3}}$ are in a natural correspondence with ordered sets that are covered by n (not necessarily distinct)* 2*-chains labelled $\widehat{\rho}_1, \dots, \widehat{\rho}_n$.*

Set up. Say we start with an n-ary algebraic relation r on $\underline{\mathbf{3}}$, with the projections denoted by $\rho_1, \dots, \rho_n\colon \mathbf{r} \to \underline{\mathbf{3}}$. We encode r as the ordered set $\mathrm{H}(\mathbf{r})$ with the 2-chains $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ labelled. (The names of the elements of $\mathrm{H}(\mathbf{r})$ can be forgotten.)

Now start with an ordered set $\mathbf{P}$ covered by n 2-chains labelled $\widehat{\rho}_1, \dots, \widehat{\rho}_n$. Create an isomorphic copy of the bounded lattice $\mathrm{K}(\mathbf{P})$: each order-preserving map $\alpha\colon \mathbf{P} \to \underset{\sim}{\mathbf{2}}$ is renamed as an n-tuple $(a_1, \dots, a_n)$, where

- $a_i = 0$, if the 2-chain $\widehat{\rho}_i$ is disjoint from $\alpha^{-1}(1)$;
- $a_i = 1$, if the 2-chain $\widehat{\rho}_i$ is contained in $\alpha^{-1}(1)$;
- $a_i = d$, if the 2-chain $\widehat{\rho}_i$ is split by $\alpha^{-1}(1)$.

This set of n-tuples forms the decoded algebraic relation on $\underline{\mathbf{3}}$.

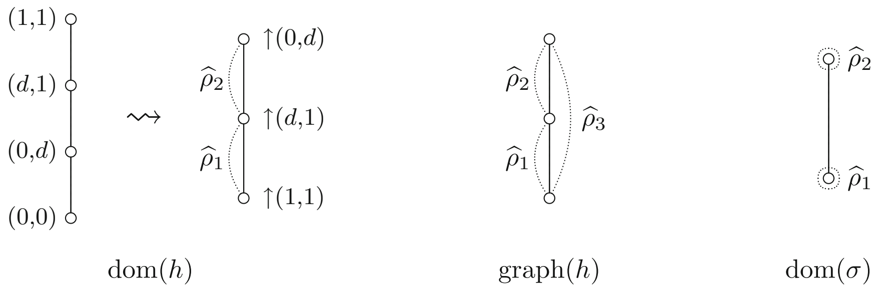

FIGURE 2. Some encoded algebraic relations on $\underset{\sim}{\mathbf{3}}$

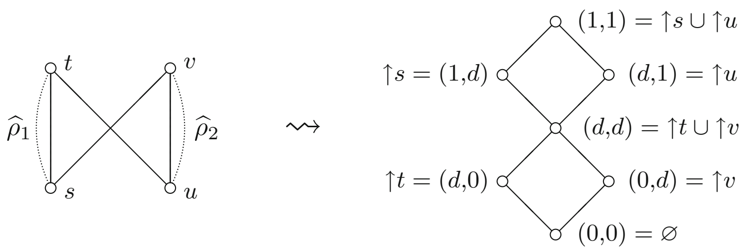

FIGURE 3. Recovering an algebraic relation on $\underset{\sim}{\mathbf{3}}$

Justification. Let r be an n-ary algebraic relation on $\underset{\sim}{\mathbf{3}}$. By Lemma 4.3, the ordered set $\mathrm{H}(\mathbf{r})$ is covered by the labelled 2-chains $\widehat{\rho}_1, \dots, \widehat{\rho}_n$. We shall check that we can recover r by applying the decoding algorithm above. We use the isomorphism $e_{\mathbf{r}} \colon \mathbf{r} \to \mathrm{KH}(\mathbf{r})$. Choose any $\alpha \colon \mathrm{H}(\mathbf{r}) \to \underset{\sim}{\mathbf{2}}$. Then $\alpha = e_{\mathbf{r}}(a)$, for some $a = (a_1, \dots, a_n) \in r$. For each $i \in \{1, \dots, n\}$, we have

$$\alpha(f \circ \rho_i) = e_{\mathbf{r}}(a)(f \circ \rho_i) = f \circ \rho_i(a) = f(a_i) \quad \text{and} \quad \alpha(g \circ \rho_i) = g(a_i).$$

Since $\widehat{\rho}_i = \{f \circ \rho_i, g \circ \rho_i\}$, it is now easy to check that the n-tuple a corresponding to α agrees with that given by the decoding algorithm.

Now let $\mathbf{P}$ be an ordered set covered by n 2-chains labelled $\widehat{\rho}_1, \dots, \widehat{\rho}_n$. We shall check that this labelled ordered set comes from an n-ary algebraic relation on $\underset{\sim}{\mathbf{3}}$. We have $\mathbf{P} \cong \mathrm{H}(\mathbf{L})$, for some $\mathbf{L} \in \mathfrak{D}_{\mathrm{fin}}$. By Lemma 4.2, there are homomorphisms $x_1, \dots, x_n \colon \mathbf{L} \to \underset{\sim}{\mathbf{3}}$ corresponding to the labelled 2-chains $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ of $\mathbf{P}$. By Lemma 4.3, these homomorphisms separate the elements of L. So the product map $\xi := x_1 \sqcap \cdots \sqcap x_n \colon \mathbf{L} \to \underset{\sim}{\mathbf{3}}^n$ is an embedding. The n-ary algebraic relation $\xi(L)$ corresponds to the given labelled ordered set. □

Figure 2 gives the encodings of three important algebraic relations on $\underset{\sim}{\mathbf{3}}$. Figure 3 gives an example of decoding an appropriately labelled ordered set. (Note that we identify a map into $\{0, 1\}$ with the pre-image of 1.)

We can easily tell whether or not an algebraic relation on $\underset{\sim}{\mathbf{3}}$ is quasi-boolean from its encoding as a labelled ordered set (see 3.3).

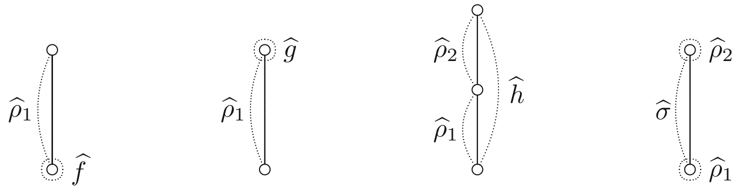

FIGURE 4. Some encoded algebraic partial operations on $\underline{\mathbf{3}}$

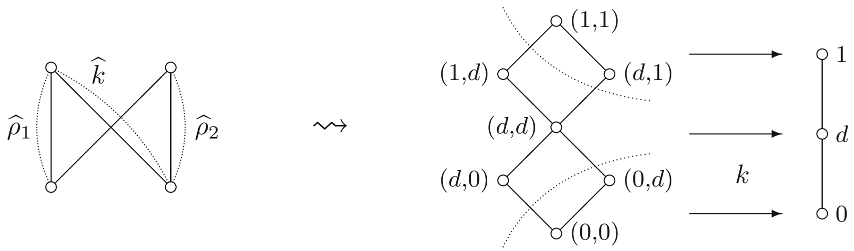

FIGURE 5. Recovering an algebraic partial operation on $\underline{\mathbf{3}}$

Lemma 4.5. *Let r be an n-ary algebraic relation on $\underline{\mathbf{3}}$, for some $n \geqslant 1$, with the projections $\rho_1, \dots, \rho_n \colon \mathbf{r} \to \underline{\mathbf{3}}$. Then the following are equivalent:*

(1) *r is quasi-boolean;*
(2) *for every increasing but not decreasing subset A of $\mathrm{H}(\mathbf{r})$, there is some $\widehat{\rho}_i$ split by A (that is, with $g \circ \rho_i \in A$ and $f \circ \rho_i \notin A$).*

Proof. Both (1) and (2) are equivalent to the following condition: for each element $a \in r$ that does not have a complement in $\mathbf{r}$, there is some i with $\rho_i(a) = d$. □

Using the previous lemma and Figure 2, it is easy to confirm that the relations $\mathrm{dom}(h)$ and $\mathrm{graph}(h)$ are quasi-boolean, while $\mathrm{dom}(\sigma)$ is not.

We can now encode algebraic partial operations on $\underline{\mathbf{3}}$ by using Correspondence 4.4 to encode their graphs. Applying Lemma 4.3, we easily obtain the following correspondence.

Correspondence 4.6. *Let $n \geqslant 1$. The n-ary algebraic partial operations k on $\underline{\mathbf{3}}$ are in a natural correspondence with ordered sets that are covered by n 2-chains labelled $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ and that also have a 2-chain labelled $\widehat{k}$.*

Figure 4 gives the encodings of the important algebraic partial operations f, g, h and σ on $\underline{\mathbf{3}}$. Figure 5 follows on from Figure 3, and gives an example of decoding an algebraic partial operation on $\underline{\mathbf{3}}$.

We want to be able to use the Full Enrichment Theorem, 2.7, to obtain new finite-level full dualities based on $\underline{\mathbf{3}}$. So we would like an easy method for checking condition 2.7(3), that is, for generating inside hom-sets. In the

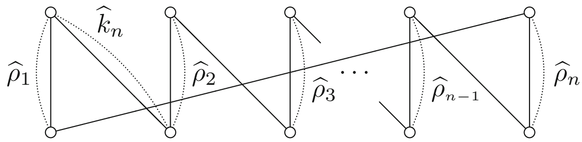

FIGURE 6. Encoding of the partial operation $k_n\colon \mathbf{r}_n \to \underline{\mathbf{3}}$

remainder of this section, we show how to describe such generation via labelled ordered sets.

The following lemma tells us how we can use Priestley duality to interpret algebraic partial operations on $\underline{\mathbf{3}}$ pointwise on hom-sets.

Generation Lemma 4.7. *Let* $\mathbf{r} \in \mathfrak{D}_{\text{fin}}$ *and* $x_1, \dots, x_n, y \in \hom(\mathbf{r}, \underline{\mathbf{3}})$, *for some* $n \geqslant 1$. *Let* $k\colon \mathbf{s} \to \underline{\mathbf{3}}$ *be an* n*-ary algebraic partial operation on* $\underline{\mathbf{3}}$, *with the projections* $\rho_1, \dots, \rho_n\colon \mathbf{s} \to \underline{\mathbf{3}}$. *Then the following are equivalent*:

(1) $y = k(x_1, \dots, x_n)$ *in* $\hom(\mathbf{r}, \underline{\mathbf{3}})$;
(2) *there is an order-preserving map* $\varphi\colon \mathrm{H}(\mathbf{s}) \to \mathrm{H}(\mathbf{r})$ *that sends the* 2*-chain* $\widehat{\rho}_i$ *onto the* 2*-chain* $\widehat{x}_i$, *for each* $i \in \{1, \dots, n\}$, *and sends the* 2*-chain* $\widehat{k}$ *onto the* 2*-chain* $\widehat{y}$.

Proof. We have $(x_1, \dots, x_n) \in s$ in $\hom(\mathbf{r}, \underline{\mathbf{3}})$ if and only if there exists a homomorphism $\psi\colon \mathbf{r} \to \mathbf{s}$ such that $\rho_i \circ \psi = x_i$, for all $i \in \{1, \dots, n\}$, in which case $\psi = x_1 \sqcap \cdots \sqcap x_n$; see [1, 8.1.2]. Using Priestley duality, we can now assume that $(x_1, \dots, x_n) \in s$ and there is an order-preserving map $\varphi\colon \mathrm{H}(\mathbf{s}) \to \mathrm{H}(\mathbf{r})$ such that $\varphi(\widehat{\rho}_i) = \widehat{x}_i$, for all $i \in \{1, \dots, n\}$, where $\varphi = \mathrm{H}(x_1 \sqcap \cdots \sqcap x_n)$. The homomorphism $k(x_1, \dots, x_n)\colon \mathbf{r} \to \underline{\mathbf{3}}$ corresponds to the morphism $\varphi \circ \mathrm{H}(k)\colon \mathrm{H}(\underline{\mathbf{3}}) \to \mathrm{H}(\mathbf{r})$, which corresponds to the 2-chain $\varphi(\widehat{k})$ in $\mathrm{H}(\mathbf{r})$. It follows that (1) $\Leftrightarrow$ (2). □

We have now developed enough theory to construct an infinite ascending chain in the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities.

Definition 4.8. Let $n \geqslant 2$ and let $k_n\colon \mathbf{r}_n \to \underline{\mathbf{3}}$ be the n-ary algebraic partial operation encoded in Figure 6. (The relation r_n encodes as the $2n$-element crown with 'vertical' covers corresponding to projections.) Define the alter ego $\underset{\sim}{\mathbf{3}}_n := \langle \{0, d, 1\}; f, g, k_n \rangle$ of $\underline{\mathbf{3}}$. The partial operation k_2 is given explicitly in Figure 5, where it is called k.

Lemma 4.9. *For* $n \geqslant 2$, *the alter ego* $\underset{\sim}{\mathbf{3}}_n$ *fully dualises* $\underline{\mathbf{3}}$ *at the finite level.*

Proof. We shall use the Full Enrichment Theorem, 2.7, to prove that $\underset{\sim}{\mathbf{3}}_n$ fully dualises $\underline{\mathbf{3}}$ at the finite level. We know that $\underset{\sim}{\mathbf{3}}_h := \langle \{0, d, 1\}; f, g, h \rangle$ fully dualises $\underline{\mathbf{3}}$ at the finite level; see 3.1. We first check that $\underset{\sim}{\mathbf{3}}_h \sqsubseteq \underset{\sim}{\mathbf{3}}_n$, by showing that h is a structural function of $\underset{\sim}{\mathbf{3}}_n$.

The domain r_h of h is conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}_n$ using f and g; see 3.1. We shall use the Generation Lemma, 4.7, to find an extension of h in

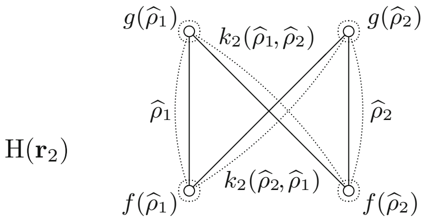

FIGURE 7. Generating the set $\hom(\mathbf{r}_2, \underline{\mathbf{3}})$ from projections

the partial clone of $\underset{\sim}{\mathbf{3}}_n$. For clarity, we use different notation for the projections $\rho_1^h, \rho_2^h \colon \mathbf{r}_h \to \underline{\mathbf{3}}$ on the domain of h and the projections $\rho_1^n, \dots, \rho_n^n \colon \mathbf{r}_n \to \underline{\mathbf{3}}$ on the domain of k_n. The two ordered sets $\mathrm{H}(\mathbf{r}_h)$ and $\mathrm{H}(\mathbf{r}_n)$ are shown in Figures 4 and 6.

There is an order-preserving map $\varphi\colon \mathrm{H}(\mathbf{r}_n) \to \mathrm{H}(\mathbf{r}_h)$ that sends $\widehat{\rho}_1^n$ onto $\widehat{\rho}_2^h$, sends each of $\widehat{\rho}_2^n, \dots, \widehat{\rho}_n^n$ onto $\widehat{\rho}_1^h$ and sends $\widehat{k_n}$ onto $\widehat{h}$. So the Generation Lemma tells us that $h = k_n(\rho_2^h, \rho_1^h, \dots, \rho_1^h)$. Thus h is a structural function of $\underset{\sim}{\mathbf{3}}_n$, whence $\underset{\sim}{\mathbf{3}}_h \sqsubseteq \underset{\sim}{\mathbf{3}}_n$.

To check condition (3) of the Full Enrichment Theorem, we want to show that each homomorphism $y\colon \mathbf{r}_n \to \underline{\mathbf{3}}$ can be generated from the projections $\rho_1, \dots, \rho_n \colon \mathbf{r}_n \to \underline{\mathbf{3}}$ using the partial operations f, g, k_n. We can use the Generation Lemma and Figure 4 to calculate generation via labelled ordered sets. We need to check that each 2-chain $\widehat{y}$ of $\mathrm{H}(\mathbf{r}_n)$ can be generated from the 2-chains $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ using the partial operations f, g, k_n. The case $n = 2$ is shown in Figure 7. □

Lemma 4.10. *For all $n \geqslant 2$, we have $\underset{\sim}{\mathbf{3}}_n \sqsubset \underset{\sim}{\mathbf{3}}_{n+1}$.*

Proof. Let $n \geqslant 2$. We can now use the fact that $\underset{\sim}{\mathbf{3}}_n$ and $\underset{\sim}{\mathbf{3}}_{n+1}$ fully dualise $\underline{\mathbf{3}}$ at the finite level. To prove that $\underset{\sim}{\mathbf{3}}_n \sqsubseteq \underset{\sim}{\mathbf{3}}_{n+1}$, we just need to show that k_n is a structural function of $\underset{\sim}{\mathbf{3}}_{n+1}$. We can do this using the Useful Lemma, 2.6, and the Generation Lemma, 4.7, by checking that each 2-chain in $\mathrm{H}(\mathbf{r}_n)$ can be generated from $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ using f, g, k_{n+1}. (For example, we have $\widehat{k_n} = k_{n+1}(\widehat{\rho}_1, \dots, \widehat{\rho}_n, \widehat{\rho}_n)$, as witnessed by the order-preserving map $\varphi\colon \mathrm{H}(\mathbf{r}_{n+1}) \to \mathrm{H}(\mathbf{r}_n)$ that sends $\widehat{\rho}_1, \dots, \widehat{\rho}_{n+1}$ onto $\widehat{\rho}_1, \dots, \widehat{\rho}_n, \widehat{\rho}_n$.)

To prove that $\underset{\sim}{\mathbf{3}}_{n+1} \not\sqsubseteq \underset{\sim}{\mathbf{3}}_n$, we again use the Useful Lemma and the Generation Lemma. To show that k_{n+1} is not a structural function of $\underset{\sim}{\mathbf{3}}_n$, we check that not every 2-chain in $\mathrm{H}(\mathbf{r}_{n+1})$ can be generated from $\widehat{\rho}_1, \dots, \widehat{\rho}_{n+1}$ using f, g, k_n. (The operations f and g generate the one-element 2-chains. To then use k_n, we need an order-preserving map from the $2n$-crown $\mathrm{H}(\mathbf{r}_n)$ to the $2(n+1)$-crown $\mathrm{H}(\mathbf{r}_{n+1})$ that sends 'vertical' covers into 'vertical' covers; but such a map must send everything into a single 'vertical' cover. Once we have developed more theory, we shall be able to give a more rigorous proof of this fact; see Example 7.5.) □

The following example comes straight from Lemmas 4.9 and 4.10.

Example 4.11. *There is an infinite ascending chain*

$$\underset{\sim}{\mathbf{3}}_2 \sqsubset \underset{\sim}{\mathbf{3}}_3 \sqsubset \underset{\sim}{\mathbf{3}}_4 \sqsubset \cdots$$

of alter egos of the bounded lattice $\underline{\mathbf{3}}$, *defined in 4.8, each of which fully dualises* $\underline{\mathbf{3}}$ *at the finite level.*

This example gives us an infinite ascending chain in the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$. We next show that, while $\mathcal{F}_{\underline{\mathbf{3}}}$ is a complete sublattice of the lattice $\mathcal{A}_{\underline{\mathbf{3}}}$ of all alter egos of $\underline{\mathbf{3}}$, it is definitely not a filter of $\mathcal{A}_{\underline{\mathbf{3}}}$.

Example 4.12. *There is an infinite ascending chain*

$$\underset{\sim}{\mathbf{3}}_2 \sqsubset \underset{\sim}{\mathbf{3}}'_2 \sqsubset \underset{\sim}{\mathbf{3}}_3 \sqsubset \underset{\sim}{\mathbf{3}}'_3 \sqsubset \underset{\sim}{\mathbf{3}}_4 \sqsubset \cdots$$

of alter egos that alternately do and do not fully dualise $\underline{\mathbf{3}}$ *at the finite level, where* $\underset{\sim}{\mathbf{3}}'_n := \langle \{0, d, 1\}; f, g, k_n, r_{n+1} \rangle$.

Proof. This claim will follow straight from Example 4.11, once we have shown that $\underset{\sim}{\mathbf{3}}'_n$ does not fully dualise $\underline{\mathbf{3}}$ at the finite level. But we observed in the proof of Lemma 4.10 that not every 2-chain in $\mathrm{H}(\mathbf{r}_{n+1})$ can be generated from the projections using f, g, k_n. As r_{n+1} is conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}'_n$, it follows by the Useful Lemma, 2.6, that $\underset{\sim}{\mathbf{3}}'_n$ does not fully dualise $\underline{\mathbf{3}}$ at the finite level. □

5. Studying $\mathcal{F}_{\underline{\mathbf{3}}}$ via coloured ordered sets

By abstracting the encoding of algebraic relations and partial operations on $\underline{\mathbf{3}}$ developed in the previous section, we shall obtain an even more powerful tool for studying the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$.

Consider the encoding of the binary relation $\mathrm{dom}(h)$, as a labelled ordered set, given at the start of Figure 2. If we swap the two labels $\widehat{\rho}_1$ and $\widehat{\rho}_2$, then the new relation that we encode is simply the converse of $\mathrm{dom}(h)$. In general, if we permute the labels $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ in the encoding of an n-ary algebraic relation r on $\underline{\mathbf{3}}$, then this simply induces the corresponding permutation in the coordinates of the relation.

Now say that, instead, we change the encoding of $\mathrm{dom}(h)$ in Figure 2 by adding an additional label $\widehat{\rho}_3$ to the 2-chain labelled $\widehat{\rho}_2$. This corresponds to making $\mathrm{dom}(h)$ into a ternary relation on $\underline{\mathbf{3}}$ by repeating its second coordinate. In general, adding or removing repeated labels in the encoding corresponds to adding or removing repeated coordinates in the relation.

Since we are only concerned with alter egos of $\underline{\mathbf{3}}$ up to structural equivalence, we do not care about permutations or repetitions of coordinates. So, when using the ordered set $\mathrm{H}(\mathbf{r})$ to encode an n-ary algebraic relation r on $\underline{\mathbf{3}}$, we can forget the labels $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ and just keep the dotted lines that mark the 2-chains corresponding to projections. We shall view these marked 2-chains as having been 'coloured'. (The dotted lines are standing in for the solid red lines used in our working notes.)

Definition 5.1. Let $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ be a non-empty structure, where both $\leqslant$ and $\lhd$ are binary relations. Then we call $\mathbf{C}$ a *coloured ordered set* if

(1) $\langle C; \leqslant \rangle$ is an ordered set such that each connected component is finite, and
(2) $\lhd$ is a subset of $\leqslant$ such that $C = \pi_1(\lhd) \cup \pi_2(\lhd)$.

We call $\leqslant$ the *order relation* of $\mathbf{C}$, and we call $\lhd$ the *colour relation* of $\mathbf{C}$ (shown in diagrams with dotted lines). A morphism of coloured ordered sets must be both order-preserving and colour-preserving.

Given a coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ and a connected component B of the order-reduct $\langle C; \leqslant \rangle$, the substructure of $\mathbf{C}$ with universe B is itself a coloured ordered set; we shall call these substructures the *connected components* of $\mathbf{C}$.

Based on Lemma 4.5, we say that a coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ is *quasi-boolean* if, for each $\leqslant$-increasing but not $\leqslant$-decreasing subset U of $\mathbf{C}$, there is some $a \lhd b$ in $\mathbf{C}$ with $a \notin U$ and $b \in U$.

Correspondence 5.2. *Let $n \geqslant 1$. The n-ary algebraic relations on $\underline{\mathbf{3}}$ are in a natural correspondence with coloured ordered sets $\langle C; \leqslant, \lhd \rangle$ such that $|\lhd| \leqslant n$.*

Set up and justification. This follows from Correspondence 4.4. Say we start with an n-ary algebraic relation r on $\underline{\mathbf{3}}$. We first form the ordered set $\mathrm{H}(\mathbf{r})$. Then we colour (rather than label) each of the 2-chains $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ corresponding to the projections $\rho_1, \dots, \rho_n \colon \mathbf{r} \to \underline{\mathbf{3}}$. Since 2-chains in $\mathrm{H}(\mathbf{r})$ correspond to elements of the order relation on $\mathrm{H}(\mathbf{r})$ and since the 2-chains $\widehat{\rho}_1, \dots, \widehat{\rho}_n$ cover $\mathrm{H}(\mathbf{r})$, we create a coloured ordered set with $|\lhd| \leqslant n$.

In the other direction, we start from a coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ such that $|\lhd| \leqslant n$. We view $\lhd$ as the set of coloured 2-chains of the ordered set $\langle C; \leqslant \rangle$. Condition 5.1(2) guarantees that the coloured 2-chains cover C. We now choose any labelling of the coloured 2-chains with $\widehat{\rho}_1, \dots, \widehat{\rho}_n$: each of the n labels must be used exactly once, and each coloured 2-chain must have at least one label. This labelled ordered set corresponds to an n-ary algebraic relation on $\underline{\mathbf{3}}$. The relation is independent of our choice of labelling, up to permutation and repetition of coordinates. □

An algebraic relation r on $\underline{\mathbf{3}}$ is *directly decomposable* if, modulo coordinate permutation, it is of the form $p \times q$, for some algebraic relations p and q on $\underline{\mathbf{3}}$. Otherwise, the relation r is *directly indecomposable*.

Lemma 5.3. *Under Correspondence 5.2 above, the directly indecomposable algebraic relations on $\underline{\mathbf{3}}$ correspond to connected coloured ordered sets.*

Proof. We shall see that this follows straight from the correspondence between products in $\mathfrak{D}_{\mathrm{fin}}$ and disjoint unions in $\mathfrak{P}_{\mathrm{fin}}$ under Priestley duality. Let r be an n-ary algebraic relation on $\underline{\mathbf{3}}$, for some $n \geqslant 1$.

First assume that r is directly decomposable. Then $\mathbf{r} \cong \mathbf{p} \times \mathbf{q}$, for algebraic relations p and q on $\underline{\mathbf{3}}$. So $\mathrm{H}(\mathbf{r}) \cong \mathrm{H}(\mathbf{p}) \mathbin{\dot\cup} \mathrm{H}(\mathbf{q})$, and therefore r encodes as a disconnected coloured ordered set, under Correspondence 5.2.

Now assume that r encodes as a disconnected coloured ordered set $\mathbf{C} \mathbin{\dot\cup} \mathbf{D}$. Let p and q be algebraic relations on $\underline{\mathbf{3}}$ that encode as $\mathbf{C}$ and $\mathbf{D}$, respectively. Then $p \times q$ encodes as $\mathbf{C} \mathbin{\dot\cup} \mathbf{D}$. So r is equivalent to $p \times q$, modulo coordinate permutation and repetition. Thus r is directly decomposable. □

We can now use coloured ordered sets to encode alter egos that fully dualise $\underline{\mathbf{3}}$ at the finite level.

Definition 5.4. Let $\mathbf{C}$ be a coloured ordered set, and let $\mathcal{C}$ denote the set of all connected components of $\mathbf{C}$. Then $\mathcal{C}$ is a set of finite coloured ordered sets. Now choose any set $R_{\mathbf{C}}$ of algebraic relations on $\underline{\mathbf{3}}$ that corresponds to the set $\mathcal{C}$, via 5.2. We define the alter ego

$$\underset{\sim}{\mathbf{3}}_{\mathbf{C}} := \big\langle \{0, d, 1\};\ \{f, g, h\} \cup \textstyle\bigcup_{r \in R_{\mathbf{C}}} \hom(\mathbf{r}, \underline{\mathbf{3}}) \big\rangle.$$

We see next that, up to structural equivalence, this alter ego is independent of our particular choice for $R_{\mathbf{C}}$.

Lemma 5.5.

(1) *For each coloured ordered set* $\mathbf{C}$, *the choice made in Definition 5.4 only affects the alter ego* $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ *up to structural equivalence.*

(2) *For each coloured ordered set* $\mathbf{C}$, *the alter ego* $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ *fully dualises* $\underline{\mathbf{3}}$ *at the finite level.*

(3) *For each alter ego* $\underset{\sim}{\mathbf{3}}$ *that fully dualises* $\underline{\mathbf{3}}$ *at the finite level, there is a coloured ordered set* $\mathbf{C}$ *such that* $\underset{\sim}{\mathbf{3}} \equiv \underset{\sim}{\mathbf{3}}_{\mathbf{C}}$.

(4) *A coloured ordered set* $\mathbf{C}$ *is quasi-boolean if and only if* $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\beta}$, *that is, if and only if* $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \not\equiv \underset{\sim}{\mathbf{3}}_{\Omega}$.

Proof. Let $\mathbf{C}$ be a coloured ordered set. We know that $\underset{\sim}{\mathbf{3}}_h$ fully dualises $\underline{\mathbf{3}}$ at the finite level (see 3.1). So it follows by Corollary 2.8 that $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ fully dualises $\underline{\mathbf{3}}$ at the finite level (independent of our choice for $R_{\mathbf{C}}$). Thus (2) holds.

To prove (1), say that we choose a different set $R'_{\mathbf{C}}$ in Definition 5.4, leading to a different alter ego $\underset{\sim}{\mathbf{3}}'_{\mathbf{C}}$. We want to show that $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \equiv \underset{\sim}{\mathbf{3}}'_{\mathbf{C}}$. We know that the two alter egos $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ and $\underset{\sim}{\mathbf{3}}'_{\mathbf{C}}$ fully dualise $\underline{\mathbf{3}}$ at the finite level, by (2). So, using the Useful Lemma, 2.6, it is enough to prove that the domain of each partial operation in the type of $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ is conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}'_{\mathbf{C}}$, and vice versa. Since the two sets $R_{\mathbf{C}}$ and $R'_{\mathbf{C}}$ correspond to the same set $\mathcal{C}$ of coloured ordered sets, they are conjunct-atomic interdefinable (using permutation and repetition of coordinates). So (1) now follows easily.

For (3), assume that $\underset{\sim}{\mathbf{3}}$ fully dualises $\underline{\mathbf{3}}$ at the finite level. Define D to be the set of all directly indecomposable algebraic relations on $\underline{\mathbf{3}}$ that are conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}$. For each $r \in D$, construct a connected coloured ordered set $\mathbf{C}_r$ corresponding to r (see 5.3). We now define the coloured ordered set $\mathbf{C} := \dot\bigcup \mathcal{C}$, where $\mathcal{C} := \{\, \mathbf{C}_r \mid r \in D \,\}$.

By construction, the set of connected components of $\mathbf{C}$ is precisely $\mathcal{C}$. So, when we are setting up the alter ego $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ in Definition 5.4, we can take $R_{\mathbf{C}} := D$. We now want to prove that $\underset{\sim}{\mathbf{3}} \equiv \underset{\sim}{\mathbf{3}}_{\mathbf{C}}$. The alter egos $\underset{\sim}{\mathbf{3}}$ and $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ fully dualise $\underline{\mathbf{3}}$ at the finite level. So, as in our proof of part (1), we can use the Useful Lemma, 2.6.

We know that f, g and h are structural functions of $\underset{\sim}{\mathbf{3}}$ (see 3.1). The domain of every other partial operation in the type of $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ belongs to $R_{\mathbf{C}} = D$, and is therefore conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}$. Thus $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \sqsubseteq \underset{\sim}{\mathbf{3}}$.

Now consider the domain r of a partial operation in the type of $\underset{\sim}{\mathbf{3}}$. The relation r is essentially a product of directly indecomposable algebraic relations on $\underline{\mathbf{3}}$. We want to show that these indecomposable factor relations are conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}$, and therefore in D. So assume that $r = p \times q$, where p is an n-ary directly indecomposable algebraic relation on $\underline{\mathbf{3}}$, for some $n \geqslant 1$. Then

$$p = \{\, (x_1, \dots, x_n) \in \{0, d, 1\}^n \mid (x_1, \dots, x_n, f(x_1), \dots, f(x_1)) \in r \,\},$$

as the algebraic relation q contains the constant tuples $(0, \dots, 0)$ and $(1, \dots, 1)$. As f is a structural function of $\underset{\sim}{\mathbf{3}}$, we now have $p \in D = R_{\mathbf{C}}$. It follows that r is conjunct-atomic definable from $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$. Thus $\underset{\sim}{\mathbf{3}} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\mathbf{C}}$.

We have shown that $\underset{\sim}{\mathbf{3}} \equiv \underset{\sim}{\mathbf{3}}_{\mathbf{C}}$, whence (3) holds. Finally, the definition of a quasi-boolean coloured ordered set ensures that (4) holds; use Lemmas 3.4 and 4.5, plus claim $(*)$ from the proof of Lemma 3.4. □

We want this encoding to reflect the structure of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities. More specifically, given coloured ordered sets $\mathbf{C}$ and $\mathbf{D}$, we want to be able to determine directly whether or not $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\mathbf{D}}$.

Definition 5.6. We shall define a binary relation $\sqsubseteq$ on the class of all coloured ordered sets. Consider coloured ordered sets $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ and $\mathbf{D}$. We shall explain the steps involved in showing that $\mathbf{C} \sqsubseteq \mathbf{D}$.

You can read '$\mathbf{C} \sqsubseteq \mathbf{D}$' as '$\mathbf{C}$ can be coloured using $\mathbf{D}$'. Roughly speaking, we want to use $\mathbf{D}$ to add colour to $\mathbf{C}$, so that $\mathbf{C}$ becomes completely coloured. Define steps (0), (1), (2), ... as follows.

- (0) Create $\mathbf{C}_0$ from $\mathbf{C}$ by strengthening the colour relation into an order. More precisely, let $\lhd_0$ be the reflexive, transitive closure of $\lhd$ on C and define the coloured ordered set $\mathbf{C}_0 := \langle C; \leqslant, \lhd_0 \rangle$.
- $(k+1)$ Find a morphism $\varphi_k \colon \mathbf{D} \to \mathbf{C}_k$. Then create $\mathbf{C}_{k+1}$ from $\mathbf{C}_k$ by colouring the image of φ_k. More precisely, let $\lhd_{k+1}$ be the transitive closure of $\lhd_k \cup \varphi_k(\leqslant_{\mathbf{D}})$ and define the coloured ordered set $\mathbf{C}_{k+1} := \langle C; \leqslant, \lhd_{k+1} \rangle$.

If it is possible to follow steps (0), (1), (2), ..., (ℓ), for some $\ell \geqslant 0$, in such a way that $\mathbf{C}$ becomes completely coloured (more precisely, so that $\lhd_\ell = \leqslant$), then $\mathbf{C} \sqsubseteq \mathbf{D}$.

In fact, to show that $\mathbf{C} \sqsubseteq \mathbf{D}$, it is clearly enough to use $\mathbf{D}$ to colour all the covering pairs in the order relation $\leqslant$ of $\mathbf{C}$.

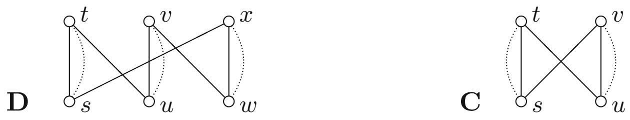

FIGURE 8. $\mathbf{C}$ can be coloured using $\mathbf{D}$

We want to show that our newly defined relation $\sqsubseteq$ on coloured ordered sets corresponds exactly to the structural-reduct relation on alter egos. But first we illustrate the definition with an example.

Example 5.7. Consider the coloured ordered sets $\mathbf{C}$ and $\mathbf{D}$ in Figure 8, which correspond to the alter egos $\underset{\sim}{\mathbf{3}}_2$ and $\underset{\sim}{\mathbf{3}}_3$ from Definition 4.8; namely, $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \equiv \underset{\sim}{\mathbf{3}}_2$ and $\underset{\sim}{\mathbf{3}}_{\mathbf{D}} \equiv \underset{\sim}{\mathbf{3}}_3$. We have already seen that $\underset{\sim}{\mathbf{3}}_2 \sqsubseteq \underset{\sim}{\mathbf{3}}_3$, in Example 4.11.

We can show that $\mathbf{C} \sqsubseteq \mathbf{D}$ by following steps (0) and (1) of the definition above, using the morphism $\varphi_0 \colon \mathbf{D} \to \mathbf{C}_0$ given by

$$s \mapsto s, \quad t \mapsto t, \quad u \mapsto u, \quad v \mapsto v, \quad w \mapsto u \quad \text{and} \quad x \mapsto v.$$

The 2-chain $\{u, t\}$ of $\mathbf{C}$ is coloured by the 2-chain $\{u, t\}$ of $\mathbf{D}$. The 2-chain $\{s, v\}$ of $\mathbf{C}$ is coloured by the 2-chain $\{s, x\}$ of $\mathbf{D}$.

Lemma 5.8. *Let $\mathbf{C}$ and $\mathbf{D}$ be coloured ordered sets. Then $\mathbf{C} \sqsubseteq \mathbf{D}$ if and only if $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\mathbf{D}}$.*

Proof. Let $\mathcal{C} = \{\, \mathbf{C}_i \mid i \in I \,\}$ and $\mathcal{D} = \{\, \mathbf{D}_j \mid j \in J \,\}$ be the sets of connected components of $\mathbf{C}$ and $\mathbf{D}$. It is easy to see from Definition 5.6 that $\mathbf{C} \sqsubseteq \mathbf{D}$ if and only if $\mathbf{C}_i \sqsubseteq \mathbf{D}$ for all $i \in I$. We know that the join in the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is given by the union of representatives; see 2.4. As $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \equiv \bigcup_{i \in I} \underset{\sim}{\mathbf{3}}_{\mathbf{C}_i}$, it follows that $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\mathbf{D}}$ if and only if $\underset{\sim}{\mathbf{3}}_{\mathbf{C}_i} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\mathbf{D}}$ for all $i \in I$. Thus we can now assume that the coloured ordered set $\mathbf{C}$ is connected and so it is also finite.

Let $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ and define $n := |\lhd|$. Choose an n-ary algebraic relation r on $\underline{\mathbf{3}}$ corresponding to $\mathbf{C}$, via 5.2, and let $\rho_1^r, \dots, \rho_n^r \colon \mathbf{r} \to \underline{\mathbf{3}}$ be the projections. Then we can identify the ordered set $\langle C; \leqslant \rangle$ with $\mathrm{H}(\mathbf{r})$, and identify the colour relation $\lhd$ with the set of 2-chains $\{\widehat{\rho}_1^r, \dots, \widehat{\rho}_n^r\}$. Each element of $\leqslant$ corresponds to a 2-chain in $\mathrm{H}(\mathbf{r})$, and therefore to a homomorphism from $\mathbf{r}$ to $\underline{\mathbf{3}}$. (We use Lemma 4.2 throughout this proof.)

As $\mathbf{C}$ corresponds to r, we can take $R_{\mathbf{C}} = \{r\}$ in Definition 5.4, giving

$$\underset{\sim}{\mathbf{3}}_{\mathbf{C}} = \big\langle \{0, d, 1\};\ \{f, g, h\} \cup \hom(\mathbf{r}, \underline{\mathbf{3}}) \big\rangle.$$

We want to show that the following are equivalent:

(1) $\underset{\sim}{\mathbf{3}}_{\mathbf{C}} \sqsubseteq \underset{\sim}{\mathbf{3}}_{\mathbf{D}}$;
(2) the set $\hom(\mathbf{r}, \underline{\mathbf{3}})$ is generated by the projections $\rho_1^r, \dots, \rho_n^r$ in $(\underset{\sim}{\mathbf{3}}_{\mathbf{D}})^r$;
(3) $\mathbf{C} \sqsubseteq \mathbf{D}$.

We know that $\underset{\sim}{\mathbf{3}}_{\mathbf{C}}$ and $\underset{\sim}{\mathbf{3}}_{\mathbf{D}}$ fully dualise $\underline{\mathbf{3}}$ at the finite level, by Lemma 5.5. So (1) $\Leftrightarrow$ (2) follows from the Useful Lemma, 2.6. To prove (2) $\Leftrightarrow$ (3), we outline how the steps in Definition 5.6 correspond to generation in $\hom(\mathbf{r}, \underline{\mathbf{3}})$.

Choose an m-ary algebraic relation $\mathbf{s}$ on $\underline{\mathbf{3}}$, for some $m \geqslant 1$, with projections $\rho_1^s, \dots, \rho_m^s : \mathbf{s} \to \underline{\mathbf{3}}$. Assume $\mathbf{s}$ corresponds to some $\mathbf{D}_j = \langle D_j; \leqslant_j, \lhd_j \rangle$ from $\mathcal{D}$. Then $\lhd_j$ corresponds to the set $\{\rho_1^s, \dots, \rho_m^s\}$ of projections, and each element of $\leqslant_j$ corresponds to an algebraic partial operation on $\underline{\mathbf{3}}$ with domain s.

Let X be a subset of $\hom(\mathbf{r}, \underline{\mathbf{3}})$ that includes the projections $\rho_1^r, \dots, \rho_n^r$. Then X corresponds to a subset $\lhd_X$ of C^2 with $\lhd \subseteq \lhd_X \subseteq \leqslant$, and $\mathbf{C}_X = \langle C; \leqslant, \lhd_X \rangle$ is a coloured ordered set.

Now consider a homomorphism $y \colon \mathbf{r} \to \underline{\mathbf{3}}$ and its corresponding 2-chain $a_y \leqslant b_y$ in $\mathbf{C}$. By the Generation Lemma, 4.7, the following are equivalent:

- the homomorphism y can be generated from X using an algebraic partial operation on $\underline{\mathbf{3}}$ with domain s;
- there is a morphism $\varphi \colon \mathbf{D}_j \to \mathbf{C}_X$ such that $(a_y, b_y) \in \varphi(\leqslant_j)$.

If the colour relation on $\mathbf{C}_X$ is reflexive, then every morphism from $\mathbf{D}_j$ to $\mathbf{C}_X$ can be extended to a morphism from $\mathbf{D}$ to $\mathbf{C}_X$. It now follows that the steps in Definition 5.6 correspond to generation of the set $\hom(\mathbf{r}, \underline{\mathbf{3}})$ in $(\underset{\sim}{\mathbf{3}}_{\mathbf{D}})^r$, where colouring a 2-chain means it has been generated. Note that generation with the endomorphisms f and g corresponds to taking the reflexive closure of the colour relation, and generation with the binary partial operation h corresponds to taking the transitive closure of the colour relation (see Figure 4). □

Definition 5.9. The structural-reduct relation $\sqsubseteq$ on the set of all alter egos of $\underline{\mathbf{3}}$ is a quasi-order. So it follows from Lemma 5.8 that our new relation $\sqsubseteq$ on the class of all coloured ordered sets is also quasi-order (with only a set of equivalence classes). We shall use $\equiv$ to denote the induced equivalence on coloured ordered sets, and $\sqsubset$ to denote the induced strict quasi-order.

We can now define $\mathcal{C}$ to be the ordered set obtained by factoring, in the natural way, the class of all coloured ordered sets equipped with the quasi-order $\sqsubseteq$. We shall usually not work with actual elements of $\mathcal{C}$ (which are equivalence classes of coloured ordered sets). We shall instead work with representatives of the elements of $\mathcal{C}$ (which are individual coloured ordered sets, but considered modulo equivalence).

We have now done nearly all the work necessary to relate our new ordered set $\mathcal{C}$ to the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities.

Corollary 5.10.

(1) *The ordered set $\mathcal{C}$ of coloured ordered sets is isomorphic to the ordered set $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities, and thus forms a doubly algebraic lattice.*
(2) *The join in $\mathcal{C}$ corresponds to the disjoint union of representatives: for coloured ordered sets $\mathbf{C}$ and $\mathbf{D}$, we have $(\mathbf{C}/{\equiv}) \vee (\mathbf{D}/{\equiv}) = (\mathbf{C} \,\dot\cup\, \mathbf{D})/{\equiv}$.*
(3) *A coloured ordered set is a representative for the top element of $\mathcal{C}$ if and only if it is not is quasi-boolean.*
(4) *A coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ is a representative for the bottom element of $\mathcal{C}$ if and only if $\leqslant$ is the reflexive, transitive closure of $\lhd$ on C.*

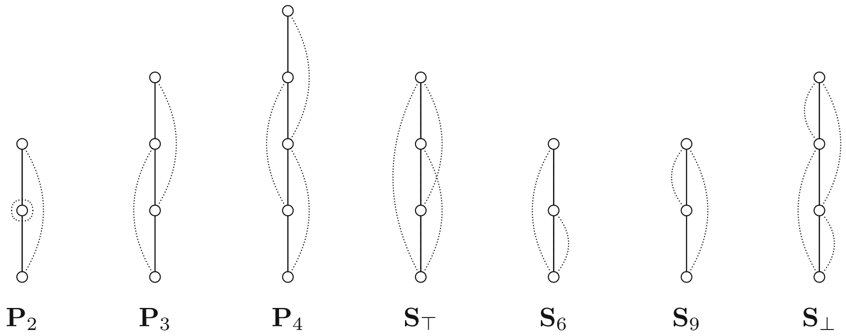

FIGURE 9. Some different coloured ordered sets

Proof. Define $\xi\colon \mathcal{C} \to \mathcal{F}_{\underline{\mathbf{3}}}$ by $\xi(\mathbf{C}/{\equiv}) := \underset{\sim}{\mathbf{3}}_{\mathbf{C}}/{\equiv}$. Then Lemmas 5.5 and 5.8 can be used to show that ξ is a well-defined order-isomorphism. The remaining claims in parts (1) and (2) now follow from 2.4, part (3) follows from Lemma 5.5(4), and part (4) follows straight from Definition 5.6. □

In contrast with joins, there is no easy way to calculate meets in the lattice $\mathcal{C}$ using representatives. We will be able to say a surprising amount about the structure of $\mathcal{C}$, considering that we will not calculate any meets.

Remark 5.11. Using Definition 5.6, we easily see that every coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd\rangle$ is equivalent to a coloured ordered set $\mathbf{C}_0 = \langle C; \leqslant, \lhd_0\rangle$ such that $\lhd_0$ is an order. (Just take $\lhd_0$ to be the reflexive, transitive closure of $\lhd$ on C.) From now on, when we specify a coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd\rangle$ via a diagram, we shall always take $\lhd$ to be an order on C; but in the diagram we will only draw in the minimum necessary coloured (that is, dotted) edges.

Example 5.12. We illustrate the power of our new correspondence by exhibiting an infinite descending chain in $\mathcal{C}$, and therefore in $\mathcal{F}_{\underline{\mathbf{3}}}$. For each $n \geqslant 2$, define the 'plait' coloured ordered set $\mathbf{P}_n := \langle \{0, 1, \dots, n\}; \leqslant_n, \lhd_n\rangle$ such that $\leqslant_n$ is the usual order on P_n and $\lhd_n$ is the reflexive, transitive closure of the binary relation

$$\{\, (i, i+2) \mid i \in \{0, \dots, n-2\}\,\}$$

on P_n. The first three plaits are drawn in Figure 9. Note that $\langle P_n; \leqslant_n\rangle$ is a chain, and that $\langle P_n; \lhd_n\rangle$ is the disjoint union of two chains.

We shall show that $\mathcal{C}$ contains the infinite descending chain

$$\mathbf{P}_2 \sqsupset \mathbf{P}_3 \sqsupset \mathbf{P}_4 \sqsupset \cdots .$$

Let $n \geqslant 2$. Using Definition 5.6, it is easy to check that $\mathbf{P}_{n+1} \sqsubseteq \mathbf{P}_n$: use the two embeddings from $\mathbf{P}_n$ into $\mathbf{P}_{n+1}$. To prove that $\mathbf{P}_n \not\sqsubseteq \mathbf{P}_{n+1}$, consider a morphism $\varphi\colon \mathbf{P}_{n+1} \to \mathbf{P}_n$. Since $|P_{n+1}| > |P_n|$, the map φ is not one-to-one. So φ collapses a covering pair in $\leqslant_{n+1}$. But this means that the range of φ is $\lhd_n$-connected in $\mathbf{P}_n$. Thus φ maps P_{n+1} into one of the two $\lhd_n$-chains

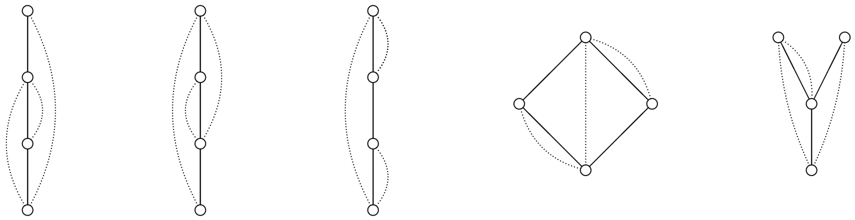

FIGURE 10. Five equivalent coloured ordered sets

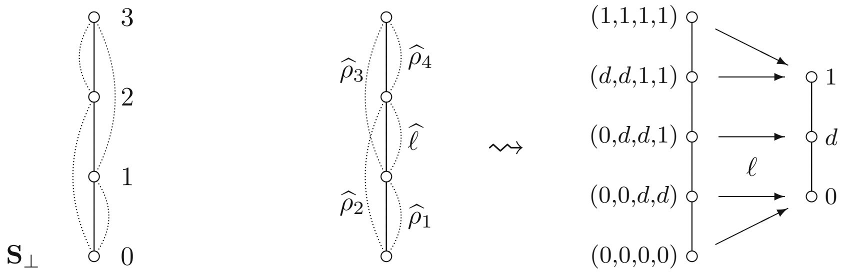

FIGURE 11. The unique atom of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$

of $\mathbf{P}_n$. But this implies that $\varphi(\leqslant_{n+1}) \subseteq \lhd_n$. So we cannot add colour to $\mathbf{P}_n$ using $\mathbf{P}_{n+1}$, whence $\mathbf{P}_n \not\sqsubseteq \mathbf{P}_{n+1}$, as required.

In fact, this infinite descending chain is an extension of our infinite ascending chain from the previous section (Example 4.11): it is easy to see that $\mathbf{P}_2 \sqsubset \mathbf{C}$, where $\mathbf{C}$ is shown in Figure 3 and corresponds to $\underset{\sim}{\mathbf{3}}_2$.

Example 5.13. We now exhibit our first antichain in $\mathcal{C}$, and therefore in $\mathcal{F}_{\underline{\mathbf{3}}}$. Define the 'spiral' coloured ordered sets $\mathbf{S}_\top$, $\mathbf{S}_6$, $\mathbf{S}_9$ and $\mathbf{S}_\perp$ as in Figure 9. It is straightforward to check that:

- $\mathbf{S}_6$ and $\mathbf{S}_9$ are non-comparable, that is, $\mathbf{S}_6 \not\sqsubseteq \mathbf{S}_9$ and $\mathbf{S}_9 \not\sqsubseteq \mathbf{S}_6$;
- $\mathbf{S}_\top$ represents the join of $\mathbf{S}_6$ and $\mathbf{S}_9$ in the lattice $\mathcal{C}$, that is, $\mathbf{S}_\top \equiv \mathbf{S}_6 \mathbin{\dot\cup} \mathbf{S}_9$;
- $\mathbf{S}_\perp$ is below the meet of $\mathbf{S}_6$ and $\mathbf{S}_9$ in $\mathcal{C}$, that is, $\mathbf{S}_\perp \sqsubseteq \mathbf{S}_6$ and $\mathbf{S}_\perp \sqsubseteq \mathbf{S}_9$.

We do not know whether or not $\mathbf{S}_\perp$ actually represents the meet of $\mathbf{S}_6$ and $\mathbf{S}_9$.

As an illustration of the care needed when studying coloured ordered sets, we display in Figure 10 a collection of five seemingly different coloured ordered sets, all of which are equivalent to $\mathbf{S}_6 \mathbin{\dot\cup} \mathbf{S}_9$.

Example 5.14. Because the lattices $\mathcal{C}$ and $\mathcal{F}_{\underline{\mathbf{3}}}$ are isomorphic, we know that the top element of $\mathcal{C}$ is completely join-irreducible (Lemma 3.4). We now show that the bottom element of $\mathcal{C}$ is completely meet-irreducible, and that the unique atom of $\mathcal{C}$ is represented by the spiral $\mathbf{S}_\perp = \langle \{0, 1, 2, 3\}; \leqslant, \lhd \rangle$ as in Figure 11. Note that $\lhd$ is taken to be an order on $S_\perp$, even though this is not shown in the diagram.

Corollary 5.10(4) tells us that $\mathbf{S}_\perp$ is not a representative for the bottom element of the lattice $\mathcal{C}$. Now let $\mathbf{D}$ be a coloured ordered set that is also not a representative for the bottom of $\mathcal{C}$. We want to show that $\mathbf{S}_\perp \sqsubseteq \mathbf{D}$.

By Corollary 5.10(4), the reflexive, transitive closure of $\lhd_{\mathbf{D}}$ is not equal to $\leqslant_{\mathbf{D}}$. Since each connected component of $\mathbf{D}$ is finite, there must be a non-coloured cover $a \leqslant_{\mathbf{D}} b$ in $\mathbf{D}$. We can define a morphism $\varphi\colon \mathbf{D} \to \mathbf{S}_\perp$ by

$$\varphi(x) = \begin{cases} 3 & \text{if } a \leqslant_{\mathbf{D}} x \text{ and } x \notin \{a, b\}, \\ 2 & \text{if } x = b, \\ 1 & \text{if } x = a, \\ 0 & \text{otherwise.} \end{cases}$$

(To check that φ is colour-preserving, note that $(1, 2)$ is the only order-related pair in $\mathbf{S}_\perp$ that is not coloured.) Since $(1, 2) = \big(\varphi(a), \varphi(b)\big) \in \varphi(\leqslant_{\mathbf{D}})$, it follows that $\mathbf{S}_\perp \sqsubseteq \mathbf{D}$. Thus the bottom element of $\mathcal{C}$ is completely meet-irreducible, and $\mathbf{S}_\perp$ represents its unique upper cover.

This simple example already tells us something about the structure of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities that might not be so easy to discover directly.

Corollary 5.15. *The bottom element of the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is completely meet-irreducible, and the alter ego $\underset{\sim}{\mathbf{3}}_\ell := \langle \{0, d, 1\}; f, g, h, \ell \rangle$ is a representative for its unique upper cover, where ℓ is the 4-ary algebraic partial operation on $\underline{\mathbf{3}}$ from Figure 11.*

Proof. The previous example (and the proof of Corollary 5.10) tells us that the bottom of $\mathcal{F}_{\underline{\mathbf{3}}}$ is completely meet-irreducible, and that the alter ego

$$\underset{\sim}{\mathbf{3}}_\perp := \langle \{0, d, 1\}; \{f, g, h\} \cup \hom(\mathbf{r}, \underline{\mathbf{3}}) \rangle$$

is a representative for its unique upper cover, where $r := \operatorname{dom}(\ell)$. Since $\underset{\sim}{\mathbf{3}}_\ell$ is a reduct of $\underset{\sim}{\mathbf{3}}_\perp$, it remains to check that $\underset{\sim}{\mathbf{3}}_\perp \sqsubseteq \underset{\sim}{\mathbf{3}}_\ell$. Using the Generation Lemma, 4.7, and Figures 4 and 11, it is easy to check that each 2-chain in $\mathrm{H}(\mathbf{r})$ can be generated from the projections using f, g, h, ℓ. So each partial operation in $\hom(\mathbf{r}, \underline{\mathbf{3}})$ has an extension in the partial clone of $\underset{\sim}{\mathbf{3}}_\ell$, as required. □

6. The lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is uncountable

In this section, we find an order-embedding from the powerset $\wp(\mathbb{N})$ into the lattice $\mathcal{C}$ of coloured ordered sets. It will then follow that the isomorphic lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities has

- cardinality $2^{\aleph_0}$,
- an uncountable antichain, and
- an uncountable chain.

Our order-embedding is built from an infinite sequence $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \ldots$ of finite coloured ordered sets.

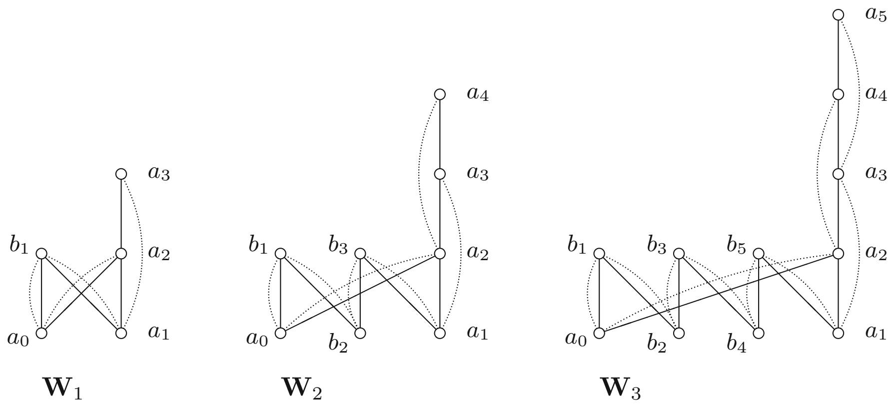

Figure 12. An infinite antichain of coloured ordered sets

Definition 6.1. Let $n \geqslant 1$. The coloured ordered set $\mathbf{W}_n = \langle W_n; \leqslant_n, \lhd_n \rangle$ is defined according to the pattern indicated in Figure 12. Note that the colour relations on $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3$ are actually the reflexive, transitive closures of those shown in the diagrams. So the colour relation $\lhd_n$ on $\mathbf{W}_n$ consists of all ordered pairs in $\leqslant_n$ except those of the form (a_i, a_j), where i and j have different parities (that is, one is odd and the other is even).

We now introduce a concept of distance in coloured ordered sets. Consider any coloured ordered set $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ such that $\lhd$ is an order on C. The colour relation $\lhd$ determines a distance function

$$\operatorname{dist}_{\lhd} \colon C^2 \to \mathbb{N} \cup \{0, \infty\},$$

where $\operatorname{dist}_{\lhd}(a, b)$ is the length of the shortest fence between a and b in the ordered set $\langle C; \lhd \rangle$. (This is the usual definition of distance in an ordered set.)

We require the following easy fact about distances in the $\mathbf{W}_n$'s.

Lemma 6.2. *Let $n \in \mathbb{N}$ and let $(c, d) \in {\leqslant_n} \backslash {\lhd_n}$. Then $\operatorname{dist}_{\lhd_n}(c, d)$ is either $2n+1$ or $2n+2$.*

Proof. As $(c, d) \in {\leqslant_n} \backslash {\lhd_n}$, we know from Definition 6.1 that $(c, d) = (a_i, a_j)$, for some i, j with different parities. Since $(a_i, a_j) \in {\leqslant_n}$, we see from Figure 12 that there are three cases:

(1) $i = 0$ and $j \geqslant 3$, with j odd;
(2) $i = 1$ and $j \geqslant 2$, with j even;
(3) $i \geqslant 2$ and $j > i$, with i and j of different parities.

In cases (1) and (2), we can see that $\operatorname{dist}_{\lhd_n}(a_i, a_j) = 2n + 1$. In case (3), we have $\operatorname{dist}_{\lhd_n}(a_i, a_j) = 2n + 2$. □

We shall now prove that the coloured ordered sets $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \ldots$ are pairwise independent, in the sense that none of them can be used to add any colour to another one.

Lemma 6.3. *Let* $\varphi\colon \mathbf{W}_m \to \mathbf{W}_n$ *be a morphism, for some* $m, n \in \mathbb{N}$ *with* $m \neq n$. *Then* $\varphi(\leqslant_m) \subseteq \lhd_n$.

Proof. Let $c \leqslant_m d$ in $\mathbf{W}_m$. We aim to show that $\varphi(c) \lhd_n \varphi(d)$ in $\mathbf{W}_n$. Since the colour relation $\lhd_n$ is transitive, we can assume that (c, d) is a covering pair in $\leqslant_m$. As φ is colour-preserving, we can also assume that $c \not\lhd_m d$. It follows that we must have $(c, d) = (a_i, a_{i+1})$, for some $i \in \{1, \dots, m+1\}$. So we now want to show that $\varphi(a_i) \lhd_n \varphi(a_{i+1})$. We consider two cases.

Case 1: $m < n$. By Lemma 6.2, we have $\operatorname{dist}_{\lhd_m}(a_i, a_{i+1}) \leqslant 2m + 2$. So

$$\operatorname{dist}_{\lhd_n}(\varphi(a_i), \varphi(a_{i+1})) \leqslant 2m + 2 < 2n + 1.$$

As φ is order-preserving, we have $\varphi(a_i) \leqslant_n \varphi(a_{i+1})$. So we can use Lemma 6.2 again to deduce that $\varphi(a_i) \lhd_n \varphi(a_{i+1})$, as required.

Case 2: $m > n$. The order-reduct of $\mathbf{W}_m$ contains the $(m+2)$-chain

$$a_1 \leqslant_m a_2 \leqslant_m \cdots \leqslant_m a_{m+2}.$$

But the longest chain in the order-reduct of $\mathbf{W}_n$ has size $n + 2 < m + 2$. Therefore $\varphi(a_j) = \varphi(a_{j+1})$, for some $j \in \{1, \dots, m+1\}$.

The index i must have the same parity as either j or $j+1$. This implies that $\operatorname{dist}_{\lhd_m}(a_i, a_j) \leqslant 1$ or $\operatorname{dist}_{\lhd_m}(a_i, a_{j+1}) \leqslant 1$. As $\varphi(a_j) = \varphi(a_{j+1})$, this gives us

$$\operatorname{dist}_{\lhd_n}(\varphi(a_i), \varphi(a_j)) \leqslant 1.$$

Similarly, we can argue that $\operatorname{dist}_{\lhd_n}(\varphi(a_{i+1}), \varphi(a_j)) \leqslant 1$. We now have

$$\begin{aligned}\operatorname{dist}_{\lhd_n}(\varphi(a_i), \varphi(a_{i+1})) &\leqslant \operatorname{dist}_{\lhd_n}(\varphi(a_i), \varphi(a_j)) + \operatorname{dist}_{\lhd_n}(\varphi(a_j), \varphi(a_{i+1})) \\ &\leqslant 2 < 2n + 1.\end{aligned}$$

By Lemma 6.2, we obtain $\varphi(a_i) \lhd_n \varphi(a_{i+1})$, as required. □

We are now ready to set up our order-embedding from $\wp(\mathbb{N})$ into $\mathcal{C}$.

Theorem 6.4. *There is an order-embedding from* $\wp(\mathbb{N})$ *into the lattice* $\mathcal{C}$, *and therefore an order-embedding from* $\wp(\mathbb{N})$ *into the isomorphic lattice* $\mathcal{F}_{\underline{\mathbf{3}}}$.

Proof. We use the coloured ordered sets $\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \dots$ from Definition 6.1 and Figure 12. Let $\mathbf{W}_\varnothing$ be the one-element coloured ordered set. For each non-empty subset S of $\mathbb{N}$, define the coloured ordered set $\mathbf{W}_S := \dot{\bigcup}_{s \in S} \mathbf{W}_s$. Define $\xi\colon \wp(\mathbb{N}) \to \mathcal{C}$ by $\xi(S) := \mathbf{W}_S/{\equiv}$. Now let $S, T \subseteq \mathbb{N}$. To prove that ξ is an order-embedding, it suffices to show that $S \subseteq T \iff \mathbf{W}_S \sqsubseteq \mathbf{W}_T$.

First assume that $S \subseteq T$. We can assume further that $S \neq \varnothing$, as $\mathbf{W}_\varnothing$ represents the bottom element of $\mathcal{C}$; see Corollary 5.10(4). We can colour $\mathbf{W}_S$ using $\mathbf{W}_T$ via a single morphism $\varphi\colon \mathbf{W}_T \to \mathbf{W}_S$ such that

- for each $s \in S$, the morphism φ maps the component $\mathbf{W}_s$ of $\mathbf{W}_T$ identically onto the corresponding component $\mathbf{W}_s$ of $\mathbf{W}_S$, and
- for each $t \in T \backslash S$, the morphism φ collapses the component $\mathbf{W}_t$ of $\mathbf{W}_T$ onto one element of $\mathbf{W}_S$.

Thus $\mathbf{W}_S \sqsubseteq \mathbf{W}_T$.

Now assume that $\mathbf{W}_S \sqsubseteq \mathbf{W}_T$ and let $s \in S$. Since the disjoint union corresponds to the join (Corollary 5.10), we must have $\mathbf{W}_s \sqsubseteq \mathbf{W}_T$. So there is a morphism $\varphi\colon \mathbf{W}_T \to \mathbf{W}_s$ that adds colour to $\mathbf{W}_s$. So there must be some $t \in T$ such that $\varphi(\leqslant_t) \nsubseteq \lhd_s$. By Lemma 6.3, this implies that $s = t$. Thus $S \subseteq T$.

The two lattices $\mathcal{C}$ and $\mathcal{F}_{\underline{\mathbf{3}}}$ are isomorphic, by Corollary 5.10. □

Using basic facts about the powerset $\wp(\mathbb{N})$, we can deduce the following information about the lattice $\mathcal{F}_{\underline{\mathbf{3}}}$. Note that the definition of $\mathcal{F}_{\underline{\mathbf{3}}}$, as consisting of equivalence classes of alter egos of $\underline{\mathbf{3}}$, ensures that its cardinality is no more than $2^{\aleph_0}$.

Corollary 6.5.

(1) *The lattice* $\mathcal{F}_{\underline{\mathbf{3}}}$ *has cardinality* $2^{\aleph_0}$.
(2) *The lattice* $\mathcal{F}_{\underline{\mathbf{3}}}$ *contains an uncountable antichain.*
(3) *The lattice* $\mathcal{F}_{\underline{\mathbf{3}}}$ *contains an uncountable chain.*

So there are uncountably many different full dualities for $\mathfrak{D}_{\text{fin}}$ based on $\underline{\mathbf{3}}$.

7. Images of coloured-ordered-set morphisms

The next two sections build a foundation for the final section, in which we prove that the lattice $\mathcal{C}$ is not modular. In this section, we introduce the 'image' of a morphism of coloured ordered sets. Note that image is a difficult concept even for plain ordered sets, as a quotient of an ordered set is not necessarily an ordered set.

Definition 7.1. Let $\varphi\colon \mathbf{D} \to \mathbf{C}$ be a morphism of coloured ordered sets. Define the coloured ordered set $\mathbf{im}(\varphi) := \langle \varphi(D); \leqslant_\varphi, \lhd_\varphi \rangle$, where

(1) the order relation $\leqslant_\varphi$ is the transitive closure of $\varphi(\leqslant_{\mathbf{D}})$, and
(2) the colour relation $\lhd_\varphi$ is equal to $\leqslant_\varphi \cap \lhd_{\mathbf{C}}$.

(See Figure 13 for an example.) It is straightforward to check that $\mathbf{im}(\varphi)$ is indeed a coloured ordered set. There are two natural morphisms associated with $\mathbf{im}(\varphi)$:

(1) the inclusion $\iota\colon\ \mathbf{im}(\varphi) \to \mathbf{C}$ is a morphism, and so we say that $\mathbf{im}(\varphi)$ is a *weakened substructure* of $\mathbf{C}$;
(2) the morphism $\varphi\colon \mathbf{D} \to \mathbf{im}(\varphi)$ witnesses that $\mathbf{im}(\varphi) \sqsubseteq \mathbf{D}$.

Note that we take the order relation on $\mathbf{im}(\varphi)$ to be that induced by $\mathbf{D}$, not $\mathbf{C}$; this ensures that $\mathbf{im}(\varphi) \sqsubseteq \mathbf{D}$. But the colour relation on $\mathbf{im}(\varphi)$ is induced as far as possible by $\mathbf{C}$.

Our non-modularity example in the final section will be built from coloured ordered sets of height 1. First consider a coloured ordered set $\mathbf{C}$ whose order-reduct is a fence of height 1. By drawing a diagram, it is easy to see that, if $\mathbf{C}$ is quasi-boolean, then all covering pairs $a \leqslant b$ in $\mathbf{C}$ must be coloured. Our next lemma generalises this fact.

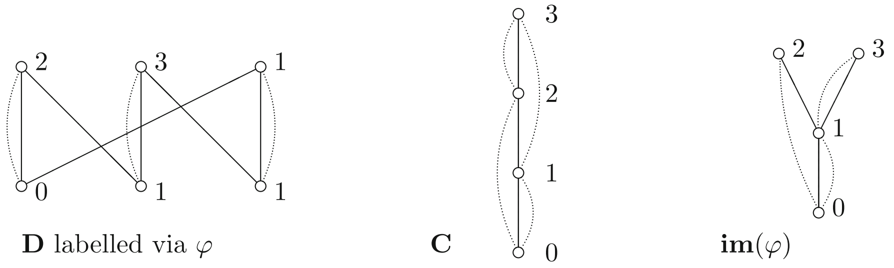

FIGURE 13. The image of a morphism $\varphi\colon \mathbf{D} \to \mathbf{C}$

Definition 7.2. Let $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ be a coloured ordered set. A *crown* in $\mathbf{C}$ consists of distinct elements $c_0, c_1, \ldots, c_n$ of C, for some odd $n \geqslant 3$, such that

$$c_0 \leqslant c_1 \geqslant c_2 \leqslant \cdots \geqslant c_{n-1} \leqslant c_n \geqslant c_0$$

in $\mathbf{C}$, and these are the only order-comparabilities in $\mathbf{C}$ between these elements.

Lemma 7.3. *Let $\mathbf{C} = \langle C; \leqslant, \lhd \rangle$ be a quasi-boolean coloured ordered set. Let $a \leqslant b$ be a covering pair in $\mathbf{C}$, and assume that a is minimal and b is maximal in $\langle C; \leqslant \rangle$. If $a \not\lhd b$, then there is a crown in $\mathbf{C}$ containing both a and b.*

Proof. Assume that there is no crown in $\mathbf{C}$ containing both a and b. We shall prove that $a \lhd b$. Our assumptions guarantee that the relation $\leqslant_\flat := \leqslant \backslash \{(a,b)\}$ is an order on C. So define the ordered set $\mathbf{C}_\flat := \langle C; \leqslant_\flat \rangle$ and let B denote the connected component of $\mathbf{C}_\flat$ containing b. Our proof now proceeds via a sequence of claims.

Claim 1: $a \notin B$. Suppose that $a \in B$. There is a minimal length fence from b to a in $\mathbf{C}_\flat$. As b is maximal and a is minimal, this fence will be of the form

$$b =: c_1 \geqslant_\flat c_2 \leqslant_\flat c_3 \geqslant_\flat c_4 \leqslant_\flat \cdots \geqslant_\flat c_{n-1} \leqslant_\flat c_n \geqslant_\flat c_0 := a,$$

where the elements $c_0, \ldots, c_n$ of C are distinct and there are no extra comparabilities between these elements in $\mathbf{C}_\flat$. Since $a \not\leqslant_\flat b$ in $\mathbf{C}_\flat$, we must have $n \geqslant 3$. Now, as $c_0 = a \leqslant b = c_1$ in $\mathbf{C}$, this fence gives rise to a crown in $\mathbf{C}$ containing a and b, which is a contradiction.

Claim 2: B is a $\leqslant$-increasing subset of $\mathbf{C}$. Let $c \leqslant d$ in $\mathbf{C}$ with $c \in B$. There is a fence from b to c in $\mathbf{C}_\flat$. Since $c \leqslant d$ in $\mathbf{C}$, this fence will extend to a fence from b to d in $\mathbf{C}_\flat$, unless $(c,d) = (a,b)$. We have $a \notin B$, by Claim 1, and $c \in B$, by assumption. Thus $a \neq c$, whence $d \in B$.

Claim 3: B is not a $\leqslant$-decreasing subset of $\mathbf{C}$. We have $b \in B$ and $a \notin B$ with $a \leqslant b$ in $\mathbf{C}$. So B is not a $\leqslant$-decreasing subset of $\mathbf{C}$.

Claim 4: (a,b) is the only pair in $\leqslant$ that is split by B. Assume that $c \leqslant d$ in $\mathbf{C}$ with $d \in B$ and $c \notin B$. There is a fence from b to d in $\mathbf{C}_\flat$. Since $c \leqslant d$ in $\mathbf{C}$, this fence will extend to a fence from b to c in $\mathbf{C}_\flat$, unless $(c,d) = (a,b)$. As $c \notin B$, there is no fence from b to c in $\mathbf{C}_\flat$. Thus $(c,d) = (a,b)$, as required.

The result follows from the preceding claims. We showed that B is a $\leqslant$-increasing but not $\leqslant$-decreasing subset of $\mathbf{C}$. Since $\mathbf{C}$ is quasi-boolean, the set B splits a pair in $\lhd$, which is a subset of $\leqslant$. But the only pair that can be split is (a, b). Thus $a \lhd b$, as required. □

The following corollary will be very useful in the final section.

Corollary 7.4. *Let $\varphi\colon \mathbf{D} \to \mathbf{C}$ be a morphism of coloured ordered sets. Assume that $\mathbf{D}$ is quasi-boolean. Assume that the ordered set $\langle C; \leqslant_{\mathbf{C}}\rangle$ has height 1 and that $\lhd_{\mathbf{C}}$ is reflexive on C. If there are no crowns in $\mathbf{im}(\varphi)$, then $\varphi(\leqslant_{\mathbf{D}}) \subseteq \lhd_{\mathbf{C}}$.*

Proof. We will be applying the previous lemma to $\mathbf{im}(\varphi) = \langle \varphi(D); \leqslant_\varphi, \lhd_\varphi\rangle$. The coloured ordered set $\mathbf{im}(\varphi)$ must be quasi-boolean, by Corollary 5.10(3), since $\mathbf{D}$ is quasi-boolean and $\mathbf{im}(\varphi) \sqsubseteq \mathbf{D}$.

Assume that there are no crowns in $\mathbf{im}(\varphi)$ and let $(a, b) \in {\leqslant_{\mathbf{D}}}$. We want to show that $\big(\varphi(a), \varphi(b)\big) \in {\lhd_{\mathbf{C}}}$. As $\lhd_{\mathbf{C}}$ is reflexive, we can assume that $\varphi(a) \neq \varphi(b)$. Since $\mathbf{im}(\varphi)$ is a weakened substructure of $\mathbf{C}$, the ordered set $\langle \varphi(D); \leqslant_\varphi\rangle$ has height 1. So $\varphi(a) \leqslant_\varphi \varphi(b)$ is a covering pair in $\langle \varphi(D); \leqslant_\varphi\rangle$, with $\varphi(a)$ minimal and $\varphi(b)$ maximal. Since there are no crowns in $\mathbf{im}(\varphi)$, it now follows by the previous lemma that $\big(\varphi(a), \varphi(b)\big) \in {\lhd_\varphi} \subseteq {\lhd_{\mathbf{C}}}$. □

We use this corollary here to justify a previous claim about order-preserving maps between crowns.

Example 7.5. For $n \geqslant 2$, define the coloured ordered set $\mathbf{C}_n = \langle C_n; \leqslant_n, \lhd_n\rangle$ to be the $2n$-crown with 'vertical' covers coloured: the two coloured ordered sets $\mathbf{C}_3$ and $\mathbf{C}_2$ are drawn in Figure 8. (As usual, we want the colour relation to be an order, so we take $\lhd_n$ to be reflexive on C_n.) It is fairly easy to see that each $\mathbf{C}_n$ is quasi-boolean.

We now look back at the proof of Lemma 4.10. Our parenthetic claim there can be rephrased as 'Each morphism $\varphi\colon \mathbf{C}_n \to \mathbf{C}_{n+1}$ satisfies $\varphi(\leqslant_n) \subseteq \lhd_{n+1}$.' To prove this claim, we just need to show that there are no crowns in $\mathbf{im}(\varphi)$. As $\mathbf{C}_n$ is strictly smaller than $\mathbf{C}_{n+1}$, the morphism φ cannot be surjective. But the only crown in $\mathbf{C}_{n+1}$ is $\mathbf{C}_{n+1}$ itself. As $\mathbf{im}(\varphi)$ is a weakened substructure of $\mathbf{C}_{n+1}$, there are no crowns in $\mathbf{im}(\varphi)$. The corollary gives us $\varphi(\leqslant_n) \subseteq \lhd_{n+1}$, as required.

8. Join-irreducible coloured ordered sets

In the next section, we construct a pentagon $\mathcal{N}_5$ inside the lattice $\mathcal{C}$. When setting up the example, we need to show that a certain finite coloured ordered set represents a join-irreducible element of $\mathcal{C}$. We shall use a sufficient condition for join-irreducibility that we establish in this section.

We first need the following very basic lemma (which holds more generally for finite relational structures).

Lemma 8.1. *Let* $\mathbf{C}$ *and* $\mathbf{D}$ *be finite coloured ordered sets with* $|\leqslant_{\mathbf{C}}| = |\leqslant_{\mathbf{D}}|$ *and* $|\lhd_{\mathbf{C}}| = |\lhd_{\mathbf{D}}|$. *Every bijective morphism* $\varphi\colon \mathbf{D} \to \mathbf{C}$ *is an isomorphism.*

For short, we say that a coloured ordered set $\mathbf{C}$ is *join-irreducible* in $\mathcal{C}$ if $\mathbf{C}$ is a representative for a join-irreducible element of the lattice $\mathcal{C}$. Recall that the join in the lattice $\mathcal{C}$ corresponds to the disjoint union of coloured ordered sets (Corollary 5.10(2)).

Lemma 8.2. *Let* $\mathbf{C}$ *be a finite coloured ordered set. Assume that* $\lhd_{\mathbf{C}}$ *is an order on* C *and that* $\lhd_{\mathbf{C}} \neq \leqslant_{\mathbf{C}}$. *Assume further that every non-bijective morphism* $\varphi\colon \mathbf{C} \to \mathbf{C}$ *satisfies* $\varphi(\leqslant_{\mathbf{C}}) \subseteq \lhd_{\mathbf{C}}$. *Then* $\mathbf{C}$ *is join-irreducible in* $\mathcal{C}$.

Proof. We are assuming that $\lhd_{\mathbf{C}}$ is an order on C and that $\lhd_{\mathbf{C}} \neq \leqslant_{\mathbf{C}}$. So $\mathbf{C}$ is not a representative for the bottom element of $\mathcal{C}$, by Corollary 5.10(4). Now assume that $\mathbf{C} \equiv \mathbf{D} \mathbin{\dot\cup} \mathbf{E}$, for some coloured ordered sets $\mathbf{D}$ and $\mathbf{E}$. We must have $\mathbf{D} \sqsubseteq \mathbf{C}$ and $\mathbf{E} \sqsubseteq \mathbf{C}$. So it suffices to show that $\mathbf{C} \sqsubseteq \mathbf{D}$ or $\mathbf{C} \sqsubseteq \mathbf{E}$.

The coloured ordered set $\mathbf{C}$ is not completely coloured (the order $\lhd_{\mathbf{C}}$ is a proper subset of $\leqslant_{\mathbf{C}}$). Since $\mathbf{C} \sqsubseteq \mathbf{D} \mathbin{\dot\cup} \mathbf{E}$, we can use Definition 5.6 to obtain a morphism $\psi\colon \mathbf{D} \mathbin{\dot\cup} \mathbf{E} \to \mathbf{C}$ that adds colour to $\mathbf{C}$. Without loss of generality, we can assume that the morphism $\varphi := \psi{\restriction}_D\colon \mathbf{D} \to \mathbf{C}$ adds colour to $\mathbf{C}$, that is, that $\varphi(\leqslant_{\mathbf{D}}) \nsubseteq \lhd_{\mathbf{C}}$.

Define the image $\mathbf{im}(\varphi) = \langle \varphi(D); \leqslant_\varphi, \lhd_\varphi \rangle$ as in Definition 7.1. The colour relation $\lhd_\varphi = \leqslant_\varphi \cap \lhd_{\mathbf{C}}$ is an order on $\varphi(D)$, since $\lhd_{\mathbf{C}}$ is an order on C. By our choice of φ, we have

$$\varphi(\leqslant_{\mathbf{D}}) \nsubseteq \lhd_{\mathbf{C}} \implies \leqslant_\varphi \nsubseteq \lhd_{\mathbf{C}} \implies \leqslant_\varphi \nsubseteq \leqslant_\varphi \cap \lhd_{\mathbf{C}} = \lhd_\varphi.$$

So $\mathbf{im}(\varphi)$ is not completely coloured.

The morphism $\varphi\colon \mathbf{D} \to \mathbf{im}(\varphi)$ witnesses that $\mathbf{im}(\varphi) \sqsubseteq \mathbf{D}$. Since $\mathbf{D} \sqsubseteq \mathbf{C}$, we obtain $\mathbf{im}(\varphi) \sqsubseteq \mathbf{C}$, by transitivity. Using Definition 5.6 again, there must be some morphism $\xi\colon \mathbf{C} \to \mathbf{im}(\varphi)$ such that ξ adds colour to $\mathbf{im}(\varphi)$, that is, such that $\xi(\leqslant_{\mathbf{C}}) \nsubseteq \lhd_\varphi$. We can choose $a \leqslant_{\mathbf{C}} b$ in $\mathbf{C}$ with $\xi(a) \not\lhd_\varphi \xi(b)$.

We now have a morphism $\iota \circ \xi\colon \mathbf{C} \to \mathbf{C}$, where $\iota\colon \mathbf{im}(\varphi) \to \mathbf{C}$ is the inclusion. Since $\xi(a) \leqslant_\varphi \xi(b)$ and $\lhd_\varphi = \leqslant_\varphi \cap \lhd_{\mathbf{C}}$, we must have $\xi(a) \not\lhd_{\mathbf{C}} \xi(b)$. Thus the morphism $\iota \circ \xi\colon \mathbf{C} \to \mathbf{C}$ does not satisfy $\iota \circ \xi(\leqslant_{\mathbf{C}}) \subseteq \lhd_{\mathbf{C}}$. By assumption, this implies that $\iota \circ \xi$ is bijective. Using Lemma 8.1, it follows that $\iota \circ \xi\colon \mathbf{C} \to \mathbf{C}$ is an automorphism, and therefore follows that $\xi\colon \mathbf{C} \to \mathbf{im}(\varphi)$ is an isomorphism.

We now have $\mathbf{C} \cong \mathbf{im}(\varphi) \sqsubseteq \mathbf{D}$, and so $\mathbf{C} \sqsubseteq \mathbf{D}$, as required. □

The converse of this lemma does not hold, as shown by the example below.

Example 8.3. Define the coloured ordered sets $\mathbf{S}$ and $\mathbf{T}$ as in Figure 14. Then $\mathbf{S}$ satisfies the conditions of the previous lemma, and so is join-irreducible in $\mathcal{C}$. It is easy to check that $\mathbf{S} \equiv \mathbf{T}$, whence $\mathbf{T}$ is join-irreducible in $\mathcal{C}$. But $\mathbf{T}$ does not satisfy the conditions of the previous lemma, as witnessed by the morphism $\varphi\colon \mathbf{T} \to \mathbf{T}$ shown in Figure 14.

FIGURE 14. Different representatives for a join-irreducible in $\mathcal{C}$

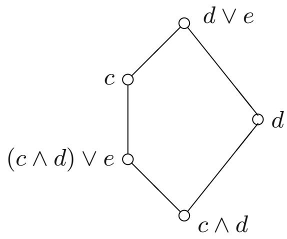

FIGURE 15. Embedding given by Lemma 9.1

The coloured ordered set $\mathbf{S}$ corresponds to the domain of the algebraic partial operation σ on $\underline{\mathbf{3}}$. It is easy to check directly that $\mathbf{S}$ is a representative for the top element of $\mathcal{C}$ and that $\mathbf{S}$ is completely join-irreducible in $\mathcal{C}$. This provides an alternative proof that the top element of $\mathcal{F}_{\underline{\mathbf{3}}}$ is completely join-irreducible; see Lemma 3.4.

9. The lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is not modular

There is already an example of a finite (size 17) lattice of finite-level full dualities $\mathcal{F}_{\underline{\mathbf{R}}}$ that is non-modular [12], where $\underline{\mathbf{R}}$ is the four-element quasi-primal algebra that Clark, Davey and Willard [2] used to create the first full but not strong duality.

We shall finish by showing that the infinite lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ is also non-modular, by proving $\mathcal{C}$ has a sublattice isomorphic to the pentagon $\mathcal{N}_5$. We can do this without explicitly calculating any meets in the lattice $\mathcal{C}$. Instead, we use the following simple lemma (which is a modification of the fact that, in a distributive lattice, every join-irreducible is join prime).

Lemma 9.1. *Let $\mathcal{L}$ be a lattice and let $c, d, e \in L$. Assume that c is join-irreducible in $\mathcal{L}$, but that $e < c \leqslant d \vee e$ and $c \nleqslant d$. Then the lattice $\mathcal{L}$ is not modular. Indeed, the pentagon $\mathcal{N}_5$ embeds into $\mathcal{L}$ as shown in Figure 15.*

Definition 9.2. We define three coloured ordered sets $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$ in Figure 16. At the top of the figure is the usual diagram for $\mathbf{C}$. At the bottom of the figure are the covering graphs for $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$. We take the colour relations $\lhd_{\mathbf{C}}$, $\lhd_{\mathbf{D}}$ and $\lhd_{\mathbf{E}}$ to be reflexive, and therefore orders.

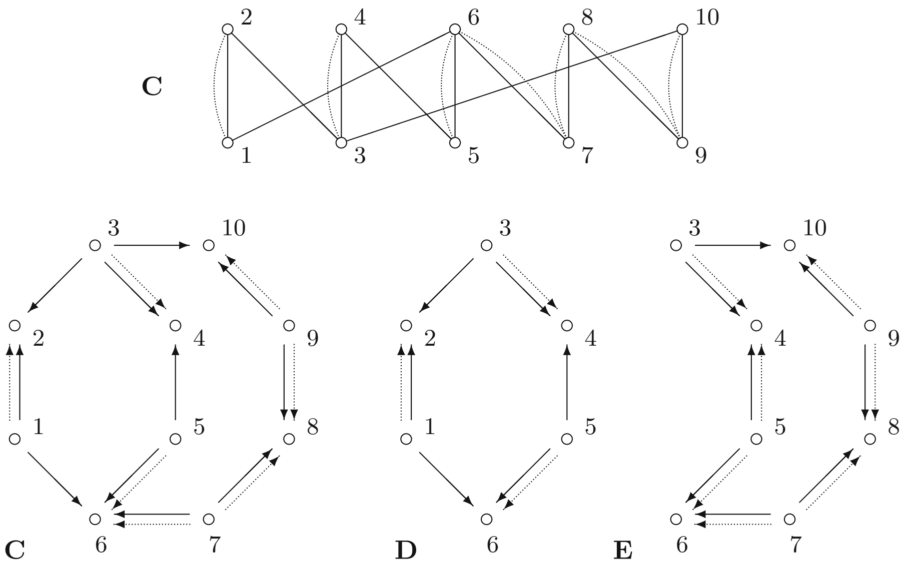

FIGURE 16. Embedding $\mathcal{N}_5$ into $\mathcal{C}$

Note that $\mathbf{D}$ is a substructure of $\mathbf{C}$, and that $\mathbf{E}$ is a colour-strengthening of a substructure of $\mathbf{C}$.

We aim to apply Lemma 9.1 in the lattice $\mathcal{C}$ using $\mathbf{C}$, $\mathbf{D}$ and $\mathbf{E}$. Since they all have height 1, we no longer have to worry about transitivity.

Lemma 9.3. *The coloured ordered sets* $\mathbf{D}$ *and* $\mathbf{E}$, *from Definition 9.2 and Figure 16, are quasi-boolean.*

Proof. The coloured ordered set $\mathbf{D}$ is quasi-boolean as it is a crown with 'vertical' covers coloured; see Example 7.5. Now let $\mathbf{E}^\flat$ denote the substructure of $\mathbf{C}$ on the set $E := C\backslash\{1,2\}$. A colour-strengthening of a quasi-boolean coloured ordered set is always quasi-boolean. So $\mathbf{E}^\flat$ is quasi-boolean (as it colour-strengthens a crown with 'vertical' covers coloured), and therefore $\mathbf{E}$ is quasi-boolean (as it colour-strengthens $\mathbf{E}^\flat$). □

Lemma 9.4. *Let* $\mathbf{C}$, $\mathbf{D}$ *and* $\mathbf{E}$ *be the coloured ordered sets specified in Definition 9.2 and Figure 16. Then* $\mathbf{E} \sqsubset \mathbf{C} \sqsubseteq \mathbf{D} \cup \mathbf{E}$ *and* $\mathbf{C} \not\sqsubseteq \mathbf{D}$.

Proof. To see that $\mathbf{E} \sqsubseteq \mathbf{C}$, we simply use the morphism $\varphi\colon \mathbf{C} \to \mathbf{E}$ given by

$$\varphi(x) = \begin{cases} 5 & \text{if } x = 1, \\ 4 & \text{if } x = 2, \\ x & \text{otherwise.} \end{cases}$$

We next show that $\mathbf{C} \not\sqsubseteq \mathbf{E}$. Let $\psi\colon \mathbf{E} \to \mathbf{C}$ be a morphism. It is enough to prove that ψ does not add colour to $\mathbf{C}$, that is, that $\psi(\leqslant_{\mathbf{E}}) \subseteq \lhd_{\mathbf{C}}$. We know that $\mathbf{E}$ is quasi-boolean, by Lemma 9.3. So, using Corollary 7.4, it is enough to prove that there are no crowns in $\mathbf{im}(\psi)$.

Recall that $\mathbf{im}(\psi)$ is a weakened substructure of $\mathbf{C}$. So by looking at the covering graph for $\mathbf{C}$ in Figure 16, we can see that, if there is a crown in $\mathbf{im}(\psi)$, then it must also be a crown in $\mathbf{C}$. There are only three crowns in $\mathbf{C}$, each of which has at least two non-coloured covers. As $\psi\colon \mathbf{E} \to \mathbf{im}(\psi)$ is colour-preserving, a non-coloured cover in $\mathbf{im}(\psi)$ must come from a non-coloured cover in $\mathbf{E}$. But $\mathbf{E}$ has only one non-coloured cover, and so $\psi(\leqslant_{\mathbf{E}})$ cannot include all the order-comparabilities in any of the crowns in $\mathbf{C}$. Thus there are no crowns in $\mathbf{im}(\psi)$, and so we have shown $\mathbf{C} \not\sqsubseteq \mathbf{E}$. It now follows that $\mathbf{E} \sqsubset \mathbf{C}$.

It is easy to see that $\mathbf{C} \sqsubseteq \mathbf{D} \cup \mathbf{E}$: in order to completely colour $\mathbf{C}$, we first use the embedding from $\mathbf{D}$ into $\mathbf{C}$ to construct $\mathbf{C}_1$, and then use the embedding from $\mathbf{E}$ into $\mathbf{C}_1$.

Since we know that $\mathbf{E} \sqsubseteq \mathbf{C}$, we can prove that $\mathbf{C} \not\sqsubseteq \mathbf{D}$ by showing that $\mathbf{E} \not\sqsubseteq \mathbf{D}$. As $\mathbf{D}$ is a 6-crown and $\mathbf{E}$ is an 8-crown, there is no morphism from $\mathbf{D}$ onto $\mathbf{E}$. As $\mathbf{D}$ is quasi-boolean, by Lemma 9.3, it follows that $\mathbf{E} \not\sqsubseteq \mathbf{D}$, by Corollary 7.4. □

Lemma 9.5. *The coloured ordered set* $\mathbf{C}$*, from Definition 9.2 and Figure 16, is join-irreducible in the lattice* $\mathcal{C}$*.*

Proof. We shall use Lemma 8.2. Let $\varphi\colon \mathbf{C} \to \mathbf{C}$ be a morphism such that $\varphi(\leqslant_{\mathbf{C}}) \not\subseteq \lhd_{\mathbf{C}}$. We shall show that φ is the identity id_C, and so is bijective. Our arguments will be based around the picture of the covering graph of $\mathbf{C}$ given in Figure 16.

Define three substructures $\mathbf{D}$, $\mathbf{E}^\flat$ and $\mathbf{F}$ of $\mathbf{C}$, with the underlying sets

$$D := \{1, 2, 3, 4, 5, 6\}, \quad E^\flat := C\backslash\{1, 2\} \quad \text{and} \quad F := C\backslash\{4, 5\}.$$

Then $\mathbf{D}$ is as shown in Figure 16, and $\mathbf{E}^\flat$ is a weakening of the coloured ordered set $\mathbf{E}$ shown in Figure 16. The coloured ordered set $\mathbf{C}$ contains exactly three crowns, on the sets D, $E^\flat$ and F. Note that both $\mathbf{D}$ and $\mathbf{E}^\flat$ are quasi-boolean (see the proof of Lemma 9.3).

We start by considering the restriction $\varphi{\restriction}_{E^\flat}\colon \mathbf{E}^\flat \to \mathbf{C}$. The image $\mathbf{im}(\varphi{\restriction}_{E^\flat})$ cannot include the crowns $\mathbf{D}$ or $\mathbf{F}$, as they have more non-coloured covers than $\mathbf{E}^\flat$. So the only crown that $\mathbf{im}(\varphi{\restriction}_{E^\flat})$ can possibly contain is $\mathbf{E}^\flat$ itself.

Case 1: $\mathbf{im}(\varphi{\restriction}_{E^\flat})$ *contains the crown* $\mathbf{E}^\flat$. It follows that $\varphi{\restriction}_{E^\flat}$ is an automorphism of $\mathbf{E}^\flat$, using Lemma 8.1. By inspecting the covering graph of $\mathbf{C}$ in Figure 16, we can see that $\mathrm{id}_{E^\flat}$ is the only automorphism of $\mathbf{E}^\flat$. Thus $\varphi{\restriction}_{E^\flat} = \mathrm{id}_{E^\flat}$, and so φ fixes 3 and 6. By a further inspection of the covering graph of $\mathbf{C}$, we can see that this forces $\varphi\colon \mathbf{C} \to \mathbf{C}$ to fix 1 and 2. Hence $\varphi = \mathrm{id}_C$.

Case 2: $\mathbf{im}(\varphi{\restriction}_{E^\flat})$ *does not contain any crowns.* It follows by Corollary 7.4 that $\varphi(\leqslant_{\mathbf{E}^\flat}) \subseteq \lhd_{\mathbf{C}}$, and therefore $\mathbf{im}(\varphi{\restriction}_{E^\flat})$ is completely coloured. Now consider the restriction $\varphi{\restriction}_D\colon \mathbf{D} \to \mathbf{C}$. As $\leqslant_{\mathbf{C}} = \leqslant_{\mathbf{D}} \cup \leqslant_{\mathbf{E}^\flat}$ and we started by assuming that $\varphi(\leqslant_{\mathbf{C}}) \nsubseteq \lhd_{\mathbf{C}}$, the image of $\varphi{\restriction}_D$ is not completely coloured. By Corollary 7.4, the image $\mathbf{im}(\varphi{\restriction}_D)$ must contain a crown. But $\mathbf{D}$ is strictly smaller than the other crowns $\mathbf{E}^\flat$ and $\mathbf{F}$ of $\mathbf{C}$, and so $\mathbf{im}(\varphi{\restriction}_D)$ contains the crown $\mathbf{D}$. Thus $\varphi{\restriction}_D$ is an automorphism of $\mathbf{D}$, again by Lemma 8.1. Since $(5,4) \in \leqslant_{\mathbf{D}} \backslash \lhd_{\mathbf{D}}$, this implies that $(\varphi(5), \varphi(4)) \in \leqslant_{\mathbf{D}} \backslash \lhd_{\mathbf{D}} \subseteq \leqslant_{\mathbf{C}} \backslash \lhd_{\mathbf{C}}$. But we also have $(5,4) \in \leqslant_{\mathbf{E}^\flat}$, and therefore $(\varphi(5), \varphi(4)) \in \varphi(\leqslant_{\mathbf{E}^\flat}) \subseteq \lhd_{\mathbf{C}}$, which is a contradiction. So this case cannot occur. □

We have shown that the coloured ordered sets **C**, **D** and **E** from Figure 16 satisfy the conditions of Lemma 9.1, and so we obtain our final result.

Theorem 9.6. *The lattice $\mathcal{C}$ of coloured ordered sets is not modular, and therefore the isomorphic lattice $\mathcal{F}_{\underline{\mathbf{3}}}$ of finite-level full dualities is not modular.*

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