STANDARD TOPOLOGICAL QUASI-VARIETIES

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Abstract. This study addresses a problem which lies at the confluence of algebra, topology and mathematical logic. It is motivated by the theory of natural dualities, which provides a tight connection between a quasi-variety $\boldsymbol{\mathcal{A}}$ and a topological quasi-variety $\boldsymbol{\mathcal{X}}$. We introduce the notion of a **standard topological quasi-variety** and initiate a program of study to determine which choices of $\boldsymbol{\mathcal{X}}$ are standard and which are not. We say that $\boldsymbol{\mathcal{X}}$ is standard if, in an appropriate sense, there is a nice axiomatic description of its members which allows us to recognize them by looking only at their finite substructures.

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be a finite structure with operations G, partial operations H, relations R and discrete topology $\mathcal{T}$. The **topological quasi-variety** generated by $\underset{\sim}{\mathbf{M}}$ is the category $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ of isomorphic copies of topologically closed substructures of direct powers, with the product topology, of $\underset{\sim}{\mathbf{M}}$. Interest in topological quasi-varieties stems from the fact that they arise as the duals to (algebraic) quasi-varieties under natural dualities. (See Clark and Davey [4].) A natural duality is a special kind of dual adjunction between the quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}\,\underline{\mathbf{M}}$ generated by a finite algebra $\underline{\mathbf{M}}$ and a topological quasi-variety $\boldsymbol{\mathcal{X}} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ generated by a structure $\underset{\sim}{\mathbf{M}}$ having the same underlying set as $\underline{\mathbf{M}}$. The general theory of natural dualities provides methods to produce, from the algebra $\underline{\mathbf{M}}$, a structure $\underset{\sim}{\mathbf{M}}$ that will yield a natural duality on $\boldsymbol{\mathcal{A}}$.

Often the structure $\underset{\sim}{\mathbf{M}}$ given by the general theory harbors a large and unwieldy collection of operations and relations, making it difficult to discern exactly which structures are in the category $\boldsymbol{\mathcal{X}}$ and what they look like. As a result, a major theme of duality theory is that of replacing $\underset{\sim}{\mathbf{M}}$ with a new dualising structure $\underset{\sim}{\mathbf{M}}'$ which is in some satisfactory sense simple enough to generate a comprehensible dual category $\boldsymbol{\mathcal{X}}' := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}'$. The literature reveals three different approaches that have been used to systematically identify and produce dualising structures that are acceptably simple.

(1) In [13] Davey and Werner gave a list of constructions that use existing operations and relations to generate new ones that are redundant for producing a duality. These constructions allowed them to eliminate structure from $\underset{\sim}{\mathbf{M}}$ until its operations and relations were reduced to an irredundant set. Davey, Haviar and Priestley [9] extended this list of constructions to a list that they could prove to be complete. (See [4], Chapter 9.)

(2) As a bottom up version of the previous approach, Davey and Priestley ([10], [11], [12]) gave inherent descriptions of those dualising structures from which no operation or relation could be eliminated without destroying the duality. Calling these **optimal dualities**, they gave efficient methods to produce them directly. (See [4], Chapter 8.)

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(3) Clark and Davey [3] formulated a series of different restrictions on the type of the structure $\underset{\sim}{\mathbf{M}}$ that would make the structures in $\mathfrak{X}$ tractable; for example, having no partial operations or having only unary operations. They then characterized those algebras $\underline{\mathbf{M}}$ which admit a strong duality safisfying each of these restrictions. (See [4], Chapter 6.)

While each of these approaches has proven useful, there are choices of $\underset{\sim}{\mathbf{M}}$ satisfying any one of these conditions that still generate incomprehensible topological quasi-varieties. For example, Saramago [19] and Wegener [22] have both found means to press the limits of the optimal duality method. Taking $\underline{\mathbf{M}}$ to be the lattice $\mathbf{N}_5$, the NU Duality Theorem [4] gives us a dualising structure $\underset{\sim}{\mathbf{M}}$ with 5896 binary relations. Wegener showed that $\underset{\sim}{\mathbf{M}}$ can be reduced to an optimal dualising structure $\underset{\sim}{\mathbf{M}}'$ with only 4 binary relations. While this constitutes a phenomenal reduction, it is by no means clear that there is any practical way to tell which 4 relation structures $\mathbf{X}'$ are in the category $\mathfrak{X}' := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}'$.

Examples like this suggest that whatever restrictions we make on the structure $\underset{\sim}{\mathbf{M}}$, the ultimate goal remains the same. In order for the dual structures in $\mathfrak{X}$ to be considered "comprehensible", there should be some way to tell which structured spaces of the right type are in $\mathfrak{X}$ and to say what they look like. These structures are identified by the existence of separating morphisms into $\underset{\sim}{\mathbf{M}}$.

Separation Theorem 0.1 ([4], 1.4.4) *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be a finite structure and let $\mathbf{X}$ be a compact topological structure of the same type as $\underset{\sim}{\mathbf{M}}$. Then $\mathbf{X} \in \mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ if and only if there is at least one morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$ and the following conditions hold:*

(i) *for each $x, y \in X$ where $x \neq y$, there is an $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $\alpha(x) \neq \alpha(y)$,*
(ii) *for each n-ary $h \in H$ and $(x_1, x_2, \ldots, x_n) \in X^n \setminus \mathrm{dom}(h^{\mathbf{X}})$, there is an $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin \mathrm{dom}(h^{\underset{\sim}{\mathbf{M}}})$,*
(iii) *for each n-ary $r \in R$ and $(x_1, x_2, \ldots, x_n) \in X^n \setminus r^{\mathbf{X}}$, there is an $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_n)) \notin r^{\underset{\sim}{\mathbf{M}}}$.*

While we will make frequent use of this basic fact, it does not always give us a readily verifiable condition for membership in $\mathfrak{X}$. What is really needed is an *axiomatic* description of the members of $\mathfrak{X}$. Clark and Krauss [5] formulated an infinitary version of quasi-atomic logic which allows assertions involving both algebra and topology. They proved that $\mathfrak{X}$ is always characterized as the class of models of the topological quasi-atomic theory of $\underset{\sim}{\mathbf{M}}$. While this result justifies our use of the phrase "topological quasi-variety", it has not yet led to any useful descriptions of the members of $\mathfrak{X}$.

In this study we propose a new sense in which a topological quasi-variety might be considered to be comprehensible. Our criterion begins with the Preservation Theorem [4](1.4.3), which we quickly review. By a **Boolean structure** (of type $\langle G, H, R \rangle$) we mean a structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}} \rangle$ such that

(i) $\langle X; \mathcal{T}^X \rangle$ is a Boolean space,
(ii) if $h \in G \cup H$ is n–ary, then $\mathrm{dom}(h^{\mathbf{X}})$ is a closed subset of X^n and $h^{\mathbf{X}} : \mathrm{dom}(h^X) \to X$ is continuous, and
(iii) if $r \in R$ is n–ary, then $r^{\mathbf{X}}$ is a closed subset of X^n.

By the **quasi-atomic theory** of $\underset{\sim}{\mathbf{M}}$ (or the **quasi-equational theory** of $\underset{\sim}{\mathbf{M}}$ if $R = \varnothing$), which we denote by $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$, we mean the set of quasi-atomic formulæ (atomic formulæ, neg-atomic formulæ and implications) which hold in $\underset{\sim}{\mathbf{M}}$. If Σ is a set of quasi-atomic formulæ, we denote by $\mathrm{Mod}_{\mathcal{T}}(\Sigma)$ the class of all Boolean structures which satisfy each quasi-atomic formula in Σ. (See [4], 1.4.)

Preservation Theorem 0.2 *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\,\rangle$ be a finite structure. Then every member of $\mathfrak{X}$ is a Boolean model of the quasi-atomic theory of $\underset{\sim}{\mathbf{M}}$, in symbols,*

$$\mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}} \subseteq \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})).$$

In particular, for every choice of $\underset{\sim}{\mathbf{M}}$ each member of $\mathfrak{X}$ is a Boolean structure of type $\langle G, H, R\rangle$. It will therefore be convenient to view the category of all Boolean structures with continuous homomorphisms as the umbrella category in which we will do our work.

We can now define the central notion of this study. We will say that $\mathfrak{X}$ is a **standard topological quasi-variety**, or that $\underset{\sim}{\mathbf{M}}$ is **standard**, if $\mathfrak{X}$ is exactly the class of all Boolean models of the quasi-atomic theory of $\underset{\sim}{\mathbf{M}}$, in symbols,

$$\mathfrak{X} = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})).$$

Thus $\mathfrak{X}$ is standard if and only if the necessary conditions of the Preservation Theorem for a structure to be in $\mathfrak{X}$ imply the sufficient conditions of the Separation Theorem. In fact standardness is an inherent property of $\mathfrak{X}$, independent of the particular choice of generator $\underset{\sim}{\mathbf{M}}$. For if $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}_1 = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}_2$, then, by the Preservation Theorem, $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_1) = \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_2)$. Thus $\mathfrak{X} = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_1))$ if and only if $\mathfrak{X} = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_2))$.

If $\Sigma \subseteq \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$, we will say that Σ **axiomatizes** $\mathfrak{X}$ provided that $\mathfrak{X} = \mathrm{Mod}_{\mathcal{T}}(\Sigma)$. If $\mathfrak{X}$ is axiomatizable, then it is certainly standard, and the axioms provide a description of its members. Davey and Werner [13] gave a number of examples of topological quasi-varieties which they showed to be standard by exhibiting a small finite set Σ that axiomatized them. These and other examples of standard topological quasi-varieties will be reviewed in Section 2, where we will also exhibit a list of familiar topological quasi-varieties that we prove to be non-standard.

Our thesis here is that we can claim to understand what is in $\mathfrak{X}$ whenever we can write down a set Σ of quasi-atomic axioms for $\mathfrak{X}$. Since an axiomatization exists if and only if $\mathfrak{X}$ is standard, the class $\mathfrak{X}$ must be standard in order to be "comprehensible" in this sense. But standardness offers us even more. While all past proofs of standardness have been obtained by *producing* an axiom system, the process of establishing an axiomatization of $\mathfrak{X}$ is considerably simplified if we know *in advance* that $\mathfrak{X}$ is standard. For suppose that we know $\mathfrak{X}$ is standard, and let $\Sigma \subseteq \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$. In Corollary 1.4 we will see that Σ axiomatizes $\mathfrak{X}$ provided only that each model of Σ is locally finite and each finite model of Σ is in $\mathfrak{X}$. Thus standardness tells us that we can axiomatize $\mathfrak{X}$ *without any reference to topology.* Once we have an axiomatization Σ for $\mathfrak{X}$, we can decide if a Boolean structure $\mathbf{X}$ is in $\mathfrak{X}$ by simply checking that $\mathbf{X}$ is locally finite and that each finite substructure of $\mathbf{X}$ satisfies Σ, again with no reference to topology. These observations motivate the following question.

> **Standardization Problem** *Which finite structures $\underset{\sim}{\mathbf{M}}$ generate a standard topological quasi-variety $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$?*

Taking the first steps to solve this general problem, we shall exhibit many examples of topological quasi-varieties $\mathfrak{X}$ which are standard and many which are not. The standardization problem for unary algebras is of particular interest since natural dualities with unary duals are among the most useful dualities. (See [4], Chapter 6.) As an illustration of our program, we will give a general proof that $\underset{\sim}{\mathbf{M}} = \langle M; f, \mathcal{T} \rangle$ is always standard if f is a single unary operation. Using this fact, we will show that a particular finite set Σ axiomatizes $\mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ by examining only the finite members of $\mathfrak{X}$.

1. General criteria for standardness

We continue to consider a finite structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ with operations G, partial operations H, relations R and discrete topology $\mathcal{T}$. If $\mathfrak{M}$ is a class of structures, we denote by $\mathfrak{M}_{fin}$ the class of finite members of $\mathfrak{M}$; by $\mathbb{P}^+\mathfrak{M}$ the class of direct products of members of $\mathfrak{M}$ over nonempty index sets (carrying the product topology); by $\mathbb{S}_c\mathfrak{M}$ the class of topologically closed substructures of members of $\mathfrak{M}$ (including the empty structure if G includes no distinguished constants) and by $\mathbb{I}\mathfrak{M}$ the class of isomorphic copies of members of $\mathfrak{M}$. If $\mathbf{X}$ is a Boolean structure and Σ is a set of quasi-atomic formulæ, then $\mathbf{X} \models \Sigma$ means that $\mathbf{X}$ satisfies each formula in Σ. We say that a Boolean structure $\mathbf{X} = \langle X; G, H, R, \mathcal{T} \rangle$ is **locally finite** if the partial algebra $\mathbf{X}' = \langle X; G, H \rangle$ is locally finite.

A classical theorem of Mal'cev [17] says that the quasi-variety generated by a finite algebra $\underline{\mathbf{M}}$ consists of all models of its quasi-atomic theory, that is,

$$\mathcal{A} := \mathbb{ISP}\,\underline{\mathbf{M}} = \mathrm{Mod}(\mathrm{Th}_{qa}(\underline{\mathbf{M}})).$$

This assertion remains true if, in addition to total operations, the finite structure $\underline{\mathbf{M}}$ has partial operations and/or relations (Andréka, Burmeister and Németi [1]). Since finite Boolean structures carry only the discrete topology, this tells us that every finite Boolean model of $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$ is in the topological quasi-variety generated by $\underset{\sim}{\mathbf{M}}$, that is,

$$\mathfrak{X}_{fin} := (\mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}})_{fin} = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}))_{fin}.$$

In this sense $\mathfrak{X}$ can always be thought of as being **finitely standard**. We will use this fact to give criteria for $\mathfrak{X}$ to be (fully) standard.

Lemma 1.1 *If $\mathbf{X}$ is a Boolean structure of the same type as $\underset{\sim}{\mathbf{M}}$, then the following are equivalent:*

(i) $\mathbf{X} \models \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$;
(ii) $\mathbf{X}$ *is locally finite and every finite substructure of* $\mathbf{X}$ *is in* $\mathfrak{X}$.

Proof Assume (i). Then every finite substructure of $\mathbf{X}$ is also a model of $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$. Since $\mathfrak{X}$ is finitely standard, every finite substructure of $\mathbf{X}$ is in $\mathfrak{X}$. To see that $\mathbf{X}$ is locally finite, let $\mathbf{X}'$ and $\underset{\sim}{\mathbf{M}}'$ be obtained from $\mathbf{X}$ and $\underset{\sim}{\mathbf{M}}$, respectively, by deleting their relations and topologies. We will show that $\mathbf{X}'$ is locally finite. By [1], $\mathbf{X}' \in \mathbb{ISP}\,\underset{\sim}{\mathbf{M}}'$, so we must check that each power $(\underset{\sim}{\mathbf{M}}')^I$ of $\underset{\sim}{\mathbf{M}}'$ is locally finite. Consider a finite set $Z \subseteq M^I$ where $|Z| = n$. For $i, j \in I$, define $i \equiv j$ if $z(i) = z(j)$ for all $z \in Z$. The equivalence $\equiv$ partitions I into at most $|M|^n$ classes, where each member of Z is constant on each class. Thus Z is contained in a substructure of $(\underset{\sim}{\mathbf{M}}')^I$ that is isomorphic to $(\underset{\sim}{\mathbf{M}}')^{|M|^n}$, and is therefore finite. This proves (ii).

Now assume (ii) and consider a quasi-atomic formula

$$\varphi := \bigwedge_{i<k} \psi_i(x_1, \dots, x_n) \to \chi(x_1, \dots, x_n)$$

in $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$. Pick $a_1, \dots, a_n \in X$ such that $\psi_i(a_1, \dots, a_n)$ is true for each $i < k$. Since $\mathbf{X}$ is locally finite it has a finite substructure $\mathbf{Y}$ containing $\{a_1, \dots, a_n\}$. By (ii) we have $\mathbf{Y} \in \mathcal{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$. By the Preservation Theorem, $\mathbf{Y} \models \varphi$ so $\chi(a_1, \dots, a_n)$ is true in $\mathbf{Y}$ and therefore also in $\mathbf{X}$. Thus $\mathbf{X} \models \varphi$.

It remains to consider a neg-atomic formula $\neg\psi(x_1, \dots, x_n) \in \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$. Let $a_1, \dots, a_n \in X$ and assume $\mathbf{X} \models \psi(a_1, \dots, a_n)$. Let $\mathbf{Y}$ be a finite substructure of $\mathbf{X}$ containing $a_1, \dots, a_n$. Then $\mathbf{Y} \models \psi(a_1, \dots, a_n)$. By (ii), we have $\mathbf{Y} \in \mathcal{X}$. By the Separation Theorem, there is a morphism $\alpha : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$. Since ψ is an atomic formula we conclude that $\underset{\sim}{\mathbf{M}} \models \psi(\alpha(a_1), \dots, \alpha(a_n))$, a contradiction. ∎

This lemma provides us with a useful criterion for standardness that will be employed throughout this study.

Corollary 1.2 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure. Then $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ is standard if and only if each locally finite Boolean structure $\mathbf{X}$ whose finite substructures are all in $\mathcal{X}$ is itself in $\mathcal{X}$.*

Since we will be interested in exhibiting examples of non-standard topological quasi-varieties, it will be helpful to explicitly state the contrapositive of Corollary 1.2.

Corollary 1.3 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure. Then $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ is non-standard if and only if there is a locally finite Boolean structure $\mathbf{X}$ such that each finite substructure of $\mathbf{X}$ is in $\mathcal{X}$ but $\mathbf{X}$ itself is not in $\mathcal{X}$.*

If we can prove *in advance* that $\mathcal{X}$ is standard, then Corollary 1.2 allows us to establish that a set $\Sigma \subseteq \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$ axiomatizes $\mathcal{X}$ without reference to topology.

Corollary 1.4 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure which generates a standard topological quasi-variety $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$. Then $\Sigma \subseteq \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$ axiomatizes $\mathcal{X}$ provided every model of Σ is locally finite and each finite model of Σ is in $\mathcal{X}$.*

We will see that the following principle can be used to establish in advance that many choices of $\mathcal{X}$ are standard without (even implicitly) finding a set of axioms for $\mathcal{X}$. Our applications will be made to total algebras, where (ii) and (iii) are not necessary.

Lemma 1.5 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure. Then $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ is standard provided that, for every $\mathbf{X} \in \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}))$,*

(i) *for each $x, y \in X$ where $x \neq y$, there is a finite $\mathbf{Y} \in \mathcal{X}$ and an $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $\alpha(x) \neq \alpha(y)$,*
(ii) *for each n-ary $h \in H$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus \mathrm{dom}(h^{\mathbf{X}})$, there is a finite $\mathbf{Y} \in \mathcal{X}$ and an $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin \mathrm{dom}(h^{\mathbf{Y}})$,*
(iii) *for each n-ary $r \in R$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus r^{\mathbf{X}}$, there is a finite $\mathbf{Y} \in \mathcal{X}$ and an $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin r^{\mathbf{Y}}$.*

Proof We show that $\mathbf{X} \in \mathfrak{X}$ by using each hypothesis to verify the corresponding part of the Separation Theorem 0.1. In each case, let $F = \{x_1, x_2, \dots\}$ be the finite set of elements of $\mathbf{X}$ that must be separated and let $\alpha : \mathbf{X} \to \mathbf{Y}$ be the given separating morphism. Since $\mathbf{Y} \in \mathfrak{X}$ there is, by the Separation Theorem, a separating morphism $\beta : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ and consequently $\beta \circ \alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ provides the required separation. ∎

2. Standard and non-standard examples

In order to provide a meaningful context for our study, we examine a number of familiar topological quasi-varieties to see which are standard and which are not. We begin by giving a list of examples known to be standard. In each case, we give a reference to the original source and (if applicable) to an item in [4], where we refer the reader for details. These proofs of standardness are all established by listing a set Σ of axioms satisfied by the generator $\underset{\sim}{\mathbf{M}}$ and then showing that every Boolean model $\mathbf{X}$ of Σ has the required separating morphisms into $\underset{\sim}{\mathbf{M}}$.

Example 2.1 The category $\mathfrak{X}$ of **Boolean spaces** $\mathbf{X} = \langle X; \mathcal{T} \rangle$ is obtained by taking $\Sigma = \varnothing$. If M is any finite set with more than one element, then $\underset{\sim}{\mathbf{M}} := \langle M; \mathcal{T} \rangle$ will serve as a generator for $\mathfrak{X}$, and $\mathfrak{X}$ is strongly dual to the variety generated by any primal algebra. (See Stone [20], [4]: 4.1.2; Hu [15], [4]: 4.1.1.)

Example 2.2 The category of **pointed Boolean spaces** $\mathbf{X} = \langle X; 0, \mathcal{T} \rangle$ is again obtained by taking $\Sigma = \varnothing$. For every prime p, this category is generated by $\underset{\sim}{\mathbf{GF}}(p) := \langle \{0, 1, \dots, p-1\}; 0, \mathcal{T} \rangle$ and is strongly dual to the quasi-variety generated by the field $\underline{\mathbf{GF}}(p) := \langle \{0, 1, \dots, p-1\}; +, \cdot, 0 \rangle$. (See [4]: 4.1.3.)

Example 2.3 The category of **Boolean rectangular bands** $\mathbf{X} = \langle X; *, \mathcal{T} \rangle$ is axiomatized by

$$\Sigma = \{x * (y * z) \approx (x * y) * z, \quad x * y \approx y * x \implies x \approx y\}.$$

It is generated by the six element band $\underset{\sim}{\mathbf{2}} \times \underset{\sim}{\mathbf{3}}$ and is strongly dual to the quasi-variety generated by the ring with identity $\underline{\mathbf{Z}}_6$. (See [4]: 4.2.6.)

Example 2.4 The category of **Boolean abelian groups of exponent** m consists of all Boolean structures $\mathbf{X} = \langle X; +, 0, \mathcal{T} \rangle$ satisfying the axioms

$$\Sigma = \{x + (y + z) \approx (x + y) + z, \quad x + y \approx y + x, \quad x + 0 \approx x, \quad mx \approx 0\}.$$

This category is generated by the cyclic group of order m and is strongly dual to the variety of all abelian groups of exponent m. (See [13]; [4]: 4.4.2.)

Example 2.5 The category of **Boolean vector spaces over** F, where F is a finite field, is axiomatized by the usual vector space axioms. It is generated by $\underset{\sim}{\mathbf{F}}$, the one dimensional vector space over F, and is strongly dual to the variety of all vector spaces over F. (See [13]; [4]: 4.4.4.)

Example 2.6 The category of **Boolean meet semilattices with 1** consists of all Boolean structures $\mathbf{X} = \langle X; \cdot, 1, \mathcal{T} \rangle$ satisfying the axioms

$$\Sigma = \{x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \quad x^2 \approx x, \quad x \cdot y \approx y \cdot x, \quad x \cdot 1 \approx x\}$$

and is strongly dual to the variety of all meet semilattices with 1. (See [14]; [4]: 4.4.7.)

Example 2.7 A large class of standard topological quasi-varieties is given in Davey and Werner [13]: the duals of varieties generated by **quasi-primal algebras**. Let $\underline{\mathbf{M}}$ be any finite algebra, and define H to be the inverse semigroup of all isomorphisms between subalgebras of $\underline{\mathbf{M}}$ together with the empty map. A **Boolean H-space** is defined to be a Boolean space $\mathbf{X} = \langle X; E, H, \mathcal{T} \rangle$ containing a set E of constants corresponding to one element subalgebras of $\underline{\mathbf{M}}$ and acted on by H in a way that satisfies a set Σ of quasi-equations listed in [5]. The authors show that the category of Boolean H-spaces is strongly dual to the variety generated by the quasi-primal algebra obtained from $\underline{\mathbf{M}}$ by adding the ternary discriminator operation. (See also [4]: 3.3.13.)

In contrast with these standard examples, most other familiar topological quasi-varieties $\mathfrak{X}$ arise as the strong dual of a quasi-variety of lattice ordered algebras. In these cases an order $\leq$ is often taken as part of the type of the structures in $\mathfrak{X}$. An ordered Boolean space $\mathbf{X} = \langle X; \leq, \mathcal{T} \rangle$ is said to be **totally order-disconnected** if, whenever $x, y \in X$ and $x \not\leq y$, there is a clopen decreasing subset of $\mathbf{X}$ containing y but not x. The category $\mathfrak{X}$ of all compact totally order-disconnected spaces was discovered by Hilary Priestley [18] as the strong dual of the variety of bounded distributive lattices, and the members of $\mathfrak{X}$ are normally referred to as **Priestley spaces**.

Notice that the definition of "totally order-disconnected" is precisely the assertion that we have the separating morphisms into the two chain $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leq, \mathcal{T} \rangle$. Thus $\mathfrak{X}$ is exactly the topological quasi-variety $\mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{2}}$. By the Preservation Theorem 0.2, each $\mathbf{X} = \langle X; \leq, \mathcal{T} \rangle$ in $\mathfrak{X}$ is a Boolean ordered space, that is, a Boolean space with a topologically closed order. This leads to a natural question. Is every Boolean ordered space a Priestley space? If so, $\mathfrak{X}$ would be a standard topological quasi-variety axiomatized by the set Σ of axioms for an ordering. In [21], Al Stralka produced an example which showed that this is not the case.

Example 2.8 *The category of Priestley spaces is not standard.* ([21]; [4]: 1.4.6.)

Proof Let $\langle C; \mathcal{T} \rangle$ be the Cantor space obtained by removing open middle thirds from the unit interval. We define the **Stralka order** $\leq$ on C by $x \leq y$ if either $x = y$ or y covers x in the usual order; for example, $\frac{1}{9} \leq \frac{2}{9}$ but $\frac{1}{9} \not\leq \frac{1}{3}$. Since $\leq$ is closed in $C \times C$, the ordered space $\mathbf{X} = \langle C; \leq, \mathcal{T} \rangle$ is a Boolean structure. Every finite subspace of $\mathbf{X}$ is a Priestley space since its topology is discrete, and $\mathbf{X}$ is locally finite since it has no operations. The only clopen decreasing subsets of $\mathbf{X}$ are initial intervals $[0, x]$ where x is the left endpoint of a deleted third. Thus $\mathbf{X}$ is not a Priestley space itself since, for example, $\frac{1}{4} \not\leq \frac{3}{4}$, but there is no clopen decreasing set containing $\frac{3}{4}$ but not $\frac{1}{4}$. By Corollary 1.3 we see that $\mathfrak{X}$ is not standard. ∎

Stralka's example tells us that, in the case of Priestley spaces, we can not replace total order-disconnectedness with quasi-atomic formulæ. Building on Stralka's example, we will show that other familiar categories dual to lattice ordered algebras are also non-standard by constructing appropriate Stralka-based Cantor-examples.

Example 2.9 *The category $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{S}}$, strongly dual to Stone algebras, is non-standard.*

Proof From Davey [7] ([4]: 4.3.7) we have $\underset{\sim}{\mathbf{S}} = \langle \{0, a, 1\}; d, \preccurlyeq, \mathcal{T} \rangle$ where $\preccurlyeq$ is the order with $1 \preccurlyeq a$ and 0 non-comparable to a and 1. For each $x \in S$, we define $d(x)$ to be the unique minimal element below x. The category $\mathfrak{X}$ consists of all Boolean structures $\mathbf{X} = \langle X; d, \preccurlyeq, \mathcal{T} \rangle$ such that $\langle X; \preccurlyeq, \mathcal{T} \rangle$ is a Priestley space and $d(x)$ is the unique minimal element below x.

To see that $\mathfrak{X}$ is non-standard, let $\mathbf{X} = \langle C; d, \preccurlyeq, \mathcal{T} \rangle$ where $\langle C; \mathcal{T} \rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order together with all pairs $(0, x)$ for $x \in C$, and $d : C \to \{0\}$ is the constant map. Clearly $\mathbf{X}$ is locally finite, d is continuous and every finite substructure of $\mathbf{X}$ is in $\mathfrak{X}$. But, as in Example 2.8, the space $\langle C; \preccurlyeq, \mathcal{T} \rangle$ is not a Priestley space so $\mathbf{X} \notin \mathfrak{X}$. Thus $\mathfrak{X}$ is non-standard by Corollary 1.3. ■

Example 2.10 *The category $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{DS}}$, strongly dual to double Stone algebras, is non-standard.*

Proof From Davey [7] ([4]: 4.3.14) we have $\underset{\sim}{\mathbf{DS}} = \langle \{0, a, b, 1\}; d, u, \preccurlyeq, \mathcal{T} \rangle$ where $\preccurlyeq$ is the order with $a \preccurlyeq b$ and no other comparabilities, where $d(x)$ is the unique minimal element below x and where $u(x)$ is the unique maximal element above x. The category $\mathfrak{X}$ consists of all Boolean structures $\mathbf{X} = \langle X; d, u, \preccurlyeq, \mathcal{T} \rangle$ such that $\langle X; \preccurlyeq, \mathcal{T} \rangle$ is a Priestley space, $d(x)$ is the unique minimal element below x and $u(x)$ is the unique maximal element above x.

To show that $\mathfrak{X}$ is non-standard, let $\mathbf{X} = \langle C; d, u, \preccurlyeq, \mathcal{T} \rangle$ where $\langle C; \mathcal{T} \rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order together with all pairs $(0, x)$ and $(x, 1)$ for $x \in X$. Let $d : C \to \{0\}$ and $u : C \to \{1\}$ be the constant maps. Clearly $\mathbf{X}$ is locally finite, d and u are continuous, and every finite substructure of $\mathbf{X}$ is in $\mathfrak{X}$. Since the space $\langle C; \preccurlyeq, \mathcal{T} \rangle$ is not a Priestley space, $\mathbf{X} \notin \mathfrak{X}$ and therefore $\mathfrak{X}$ is non-standard by Corollary 1.3. ■

Example 2.11 *The category $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{DM}}$, strongly dual to De Morgan algebras, is non-standard.*

"D".

Proof By Cornish and Fowler [6] ([4]: 4.3.16) we have $\underset{\sim}{\mathbf{DM}} = \langle \{0, a, b, 1\}; f, \preccurlyeq, \mathcal{T} \rangle$ where $\preccurlyeq$ is the non-linear lattice order with a at the top and b at the bottom, and f is the bijection fixing 0 and 1 and interchanging a and b. The category $\mathfrak{X}$ consists of all Boolean structures $\mathbf{X} = \langle X; f, \preccurlyeq, \mathcal{T} \rangle$ such that $\langle X; \preccurlyeq, \mathcal{T} \rangle$ is a Priestley space and f is an order-reversing homeomorphism of order 2.

To prove that $\mathfrak{X}$ is non-standard, let $\mathbf{X} = \langle C; f, \preccurlyeq, \mathcal{T} \rangle$ where $\langle C; \mathcal{T} \rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order and $f(x) = 1 - x$. Then $\mathbf{X}$ is locally finite, f is continuous and every finite substructure of $\mathbf{X}$ is in $\mathfrak{X}$, but the space $\langle C; \preccurlyeq, \mathcal{T} \rangle$ is not a Priestley space. Thus $\mathfrak{X}$ is non-standard by Corollary 1.3. ■

Example 2.12 *The category $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{K}}$, strongly dual to Kleene algebras, is non-standard.*

Proof Davey and Werner [13] ([4]: 4.3.10) take $\underset{\sim}{\mathbf{K}} = \langle \{0, a, 1\}; \preccurlyeq, \sim, K_0, \mathcal{T} \rangle$ and define the order $\preccurlyeq$ with $0 \preccurlyeq a$ and $1 \preccurlyeq a$, the binary relation $\sim\, = \{0, a, 1\}^2 \setminus \{(0, 1), (1, 0)\}$, and the unary relation $K_0 = \{0, 1\}$. The category $\mathcal{X}$ consists of all Boolean structures $\mathbf{X} = \langle X; \preccurlyeq, \sim, X_0, \mathcal{T} \rangle$ such that $\langle X; \preccurlyeq, \mathcal{T} \rangle$ is a Priestley space and $\mathbf{X}$ satisfies the universal axioms

(a) $x \sim x$,
(b) $x \sim y$ and $x \in X_0 \implies x \preccurlyeq y$,
(c) $x \sim y$ and $y \preccurlyeq z \implies z \sim x$.

To establish that $\mathcal{X}$ is non-standard, let $\mathbf{X} = \langle C; \preccurlyeq, \sim, X_0, \mathcal{T} \rangle$ where $\langle C; \mathcal{T} \rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order, $\sim\, = \,\preccurlyeq \cup \succcurlyeq$ and $X_0 = \{0, 1\}$. Then $\sim$ and X_0 are closed so that $\mathbf{X}$ is a Boolean structure, and $\mathbf{X}$ is locally finite. We easily check that $\mathbf{X}$ satisfies the universal axioms (a), (b) and (c) and consequently so does every substructure of $\mathbf{X}$. Thus every finite substructure of $\mathbf{X}$ is in $\mathcal{X}$. Since the space $\langle C; \preccurlyeq, \mathcal{T} \rangle$ is not a Priestley space, $\mathbf{X} \notin \mathcal{X}$ and therefore $\mathcal{X}$ is non-standard by Corollary 1.3. ∎

Example 2.13 *The category $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$, strongly dual to median algebras, is non-standard.*

Proof By Isbell [16] and Werner [23] ([4]: 4.3.4) we have $\underset{\sim}{\mathbf{M}} = \langle \{0, 1\}; *, 0, 1, \leq, \mathcal{T} \rangle$ where $\leq$ is the usual order and $*$ interchanges 0 and 1. The category $\mathcal{X}$ consists of all Boolean structures $\mathbf{X} = \langle X; *, 0, 1, \leq, \mathcal{T} \rangle$ such that $\langle X; 0, 1, \leq, \mathcal{T} \rangle$ is a bounded Priestley space, $*$ is order-reversing, interchanges 0 and 1, and satisfies $x^{**} \approx x$ and $x \leq x^* \implies x \approx 0$.

To verify that $\mathcal{X}$ is non-standard, let $\mathbf{X} = \langle C; *, 0, 1, \leq, \mathcal{T} \rangle$ where $\langle C; \mathcal{T} \rangle$ is the Cantor space, $\leq$ is the Stralka order with the pair $(\frac{1}{3}, \frac{2}{3})$ removed, and $x^* = 1 - x$. Clearly $\leq$ is closed, $*$ is continuous, and $\mathbf{X}$ is locally finite. Moreover $\mathbf{X}$ satisfies the required universal axioms and consequently every finite substructure of $\mathbf{X}$ is in $\mathcal{X}$. But the space $\langle C; \leq, \mathcal{T} \rangle$ is again not a Priestley space so $\mathbf{X} \notin \mathcal{X}$. Thus $\mathcal{X}$ is non-standard by Corollary 1.3. ∎

3. Misguided conjectures

Given a finite structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$, we would like to know whether or not its generated topological quasi-variety $\mathcal{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ is standard. The examples in the previous section all arose as natural duals of familiar quasi-varieties. As a result, information was available about them that allowed us to answer this question in those cases. But we should bear in mind that the question of standardness applies to any choice of $\underset{\sim}{\mathbf{M}}$, regardless of whether or not it generates a dual category. Based on the examples we have seen, we might venture a natural guess.

Misguided Conjecture 3.1 *A finite structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ is standard provided that it is a total or partial algebra, that is, $R = \varnothing$.*

Let $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ be the three-element chain. Then $\underline{\mathbf{M}}$ generates the (quasi-)variety of distributive lattices and has two proper endomorphisms, f and g, where $f(a) = 0$ and $g(a) = 1$. In [9] Davey, Haviar and Priestley show that $\underset{\sim}{\mathbf{M}}_{fg} = \langle \{0, a, 1\}; f, g, \mathcal{T} \rangle$ yields a duality on bounded distributive lattices relative to the generator $\underline{\mathbf{M}}$ that is neither full nor strong.

Counterexample 3.2 The total algebra $\underset{\sim}{\mathbf{M}}_{fg} = \langle \{0, a, 1\}; f, g, \mathcal{T} \rangle$ generates a non-standard topological quasi-variety $\mathfrak{X}_{fg}$.

Proof Given the set Z of all integers, the structure $\mathbf{Y}_\infty = \langle Z \cup \{\infty\}; f, g, \mathcal{T} \rangle$ is defined as follows. (See Figure 1.) For all $n \in Z$, let $f(2n) = g(2n) = 2n$, let $f(2n{+}1) = 2n$, let $g(2n{+}1) = 2n{+}2$ and let $f(\infty) = g(\infty) = \infty$. Topologically $\mathbf{Y}_\infty$ is the one point compactification of Z, where $U \subseteq Z$ is open if it is either cofinite or does not contain ∞. It is easy to check that $\mathbf{Y}_\infty$ is a Boolean structure and that every finite substructure of $\mathbf{Y}_\infty$ can be separated by morphisms into $\underset{\sim}{\mathbf{M}}_{fg}$ and is therefore in $\mathfrak{X}_{fg}$. It is also easy to check that the only continuous homomorphisms from $\mathbf{Y}_\infty$ into $\underset{\sim}{\mathbf{M}}_{fg}$ are the constant maps to 0 and to 1, and consequently $\mathbf{Y}_\infty$ is not in $\mathfrak{X}_{fg}$. By Corollary 1.3 we conclude that $\mathfrak{X}_{fg}$ is non-standard. ∎

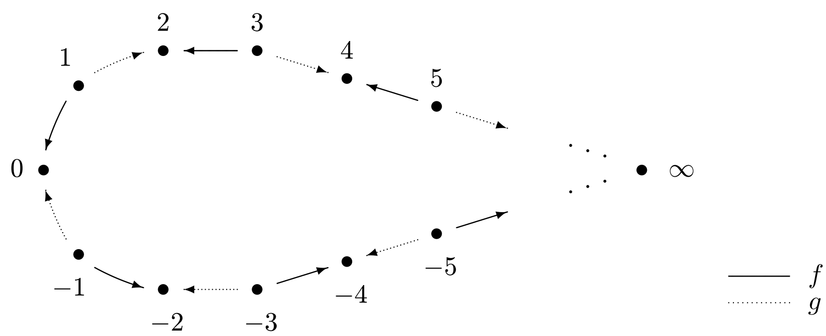

FIGURE 1. Y_∞

In fact $\underset{\sim}{\mathbf{M}}_{fg}$ has a special property that might possibly account for this aberrant behavior. The following lemma follows from Bestsennyi [2], where a complete description of the three-element unary algebras whose quasi-equational theory is finitely axiomatizable is given.

Lemma 3.3 *The quasi-equational theory* $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_{fg})$ *is not finitely axiomatizable.*

Proof Let $\Sigma \subseteq \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_{fg})$ be finite, and let n be the number of variables that occur in Σ. Let $\mathbf{Y} = \langle \{0, 1, \dots, 2n{+}1\}; f, g, \mathcal{T} \rangle$ be the discrete $(2n{+}2)$-cycle where

$$0 \overset{f}{\longleftarrow} 1 \overset{g}{\Longrightarrow} 2 \overset{f}{\longleftarrow} 3 \overset{g}{\Longrightarrow} 4 \overset{f}{\longleftarrow} \dots \overset{g}{\Longrightarrow} 2n \overset{f}{\longleftarrow} 2n+1 \overset{g}{\Longrightarrow} 0$$

and $f(2k) = g(2k) = 2k$ for all $k \leq n$. It is easy to check the only (continuous) homomorphisms from $\mathbf{Y}$ into $\underset{\sim}{\mathbf{M}}_{fg}$ are the constant maps to 0 and to 1, and consequently $\mathbf{Y}$ is not in $\mathfrak{X}_{fg}$. By Lemma 1.1 we see that $\mathbf{Y}$ is not a model of $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_{fg})$.

On the other hand it is also easy to check that every proper subalgebra $\mathbf{X}$ of $\mathbf{Y}$ can be separated by homomorphisms into $\underset{\sim}{\mathbf{M}}_{fg}$. By the Separation Theorem 0.1 we have $\mathbf{X} \in \mathfrak{X}_{fg}$, and so, by the Preservation Theorem 0.2, we conclude that $\mathbf{X}$ is a model of Σ. Since every subset of $\mathbf{Y}$ with no more than n elements generates a proper subalgebra of $\mathbf{Y}$, we see that $\mathbf{Y}$ is a model of Σ. Thus Σ does not axiomatize $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_{fg})$. ∎

Misguided Conjecture 3.4 *A finite structure $\underset{\sim}{\mathbf{M}}$ is standard provided that $R = \varnothing$ and that its quasi-equational theory $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$ is finitely axiomatizable.*

Counterexample 3.5 Consider again the three-chain $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1\rangle$. Within $\underline{\mathbf{M}} \times \underline{\mathbf{M}}$ the set $\{(0, 0), (0, 1), (1, 1)\}$ forms a bounded sublattice $\mathbf{N}$ isomorphic to $\underline{\mathbf{M}}$. We will view the isomorphism $m : \mathbf{N} \to \underline{\mathbf{M}}$ as a binary partial operation on the set $\{0, a, 1\}$. Let $\underset{\sim}{\mathbf{M}}_{fgm} = \langle \{0, a, 1\}; f, g, m, \mathcal{T}\rangle$, be the partial algebra obtained by adding m to $\underset{\sim}{\mathbf{M}}_{fg}$, and let $\mathfrak{X}_{fgm} := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}_{fgm}$ be the topological quasi-variety it generates.

Lemma 3.6 *The partial algebra $\underset{\sim}{\mathbf{M}}_{fgm}$ generates a non-standard topological quasi-variety $\mathfrak{X}_{fgm}$.*

Proof We form a Stralka-based Boolean structure $\mathbf{Z} = \langle Z; f, g, m, \mathcal{T}\rangle$ in several steps. Let $\langle C; \le, \mathcal{T}\rangle$ be the Cantor space with the Stralka order of Example 2.8. Now add to C the center point of each missing third. These are exactly the numbers between 0 and 1 whose infinite decimal expansion base–3 consists of a finite sequence of 0s and 2s followed by only 1s. Finally, bend the space into a circle by identifying 0 and 1 as a single point Q. Let C' be the image of C under this construction. We define $\langle Z; \le, \mathcal{T}\rangle$ to be the resulting Boolean space with the quotient topology and the Stralka order $\le$ on C', where now $Q \le Q$. Let $f : Z \to Z$ fix each point of C' and map each new center point to the point of C' immediately below it in the usual order. Similarly, let $g : Z \to Z$ fix each point of C' and map each new center point to the point of C' immediately above it in the usual order. Finally, let m be the binary partial operation on Z whose domain is $\le$. We take $m(x, y)$ to be x if $x = y$ and to be the midpoint of the interval $[x, y]$ if $x < y$. The following facts about the structure $\mathbf{Z}$ are now straightforward to verify.

- $\mathbf{Z}$ is a Boolean structure, that is, a Boolean space with $\le$ closed and f, g and m continuous.
- The only morphisms from $\mathbf{Z}$ to $\underset{\sim}{\mathbf{M}}_{fgm}$ are the constant maps to 0 and to 1. Thus $\mathbf{Z} \notin \mathfrak{X}_{fgm}$.
- $\mathbf{Z}$ is locally finite.
- For every finite substructure $\mathbf{Y}$ of $\mathbf{Z}$, there are separating morphisms from $\mathbf{Y}$ into $\underset{\sim}{\mathbf{M}}_{fgm}$. Thus $\mathbf{Y} \in \mathfrak{X}_{fgm}$.

It follows from Corollary 1.3 that $\mathfrak{X}_{fgm}$ is non-standard. ∎

Lemma 3.7 *The quasi-equational theory $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_{fgm})$ is finitely axiomatizable.*

Proof We take Σ to be the following (admittedly redundant) set of axioms, all of which hold in $\underset{\sim}{\mathbf{M}}_{fgm}$.

A1) $ff(x) \approx f(x) \approx gf(x), \;\; gg(x) \approx g(x) \approx fg(x)$
A2) $m(x,y) \approx m(x,y) \implies f(m(x,y)) \approx x \;\&\; g(m(x,y)) \approx y$
A3) $m(x,x) \approx m(x,x) \iff f(x) \approx x$
A4) $m(x,y) \approx m(y,x) \implies x \approx y$
A5) $m(x,y) \approx m(x,y) \;\&\; m(y,z) \approx m(y,z) \implies m(x,z) \approx m(x,z)$
A6) $m(f(x), g(x)) \approx x$
A7) $f(x) \approx g(x) \implies f(x) \approx x$
A8) $m(m(x,y),u) \approx m(m(x,y),u) \implies m(x,y) \approx x$
A9) $m(x, m(u,v)) \approx m(x, m(u,v)) \implies m(u,v) \approx u$

Recall that, in the presence of partial operations, an equation is satisfied if both sides are defined and are equal. Thus, for example, the odd looking equation $m(x,y) \approx m(x,y)$ simply asserts that $(x,y) \in \mathrm{dom}(m)$.

Let $\mathbf{X} = \langle X; f, g, m, \mathcal{T} \rangle \models \Sigma$. We will prove that $\mathbf{X} \models \mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}_{fgm})$ using Lemma 1.1. We first argue that $\mathbf{X}$ is locally finite. Let $Z \subseteq X$ be finite. Then $W := Z \cup f(Z) \cup g(Z)$ is a finite set closed under f and g by A1. Now define $Y := W \cup m(W^2 \cap \mathrm{dom}(m))$. Then Y is a finite subset of X containing Z, and Y is closed under f and g by A2 and is closed under m by A8 and A9.

Now let $\mathbf{Y}$ be a finite substructure of $\mathbf{X}$. Then $\mathbf{Y} \models \Sigma$. By A1 we see that f and g are both retractions of Y onto the same set $Y_0 \subseteq Y$. From A3, A4 and A5, we know that the domain of m is a reflexive order on Y_0 which we will denote by $\leq$.

In order to produce separating homomorphisms into $\underset{\sim}{\mathbf{M}}_{fgm}$, let U be a $\leq$–increasing subset of Y_0. For $x \in Y$ we have $f(x) \leq g(x)$ by A6, so we can define a map $\alpha_U : Y \to \{0, a, 1\}$ by

$$\alpha_U(x) = \begin{cases} 1 & \text{if } f(x) \in U \text{ and } g(x) \in U; \\ a & \text{if } f(x) \notin U \text{ and } g(x) \in U; \\ 0 & \text{if } f(x) \notin U \text{ and } g(x) \notin U. \end{cases} \tag{1}$$

We claim that $\alpha_U : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}_{fgm}$ is a morphism. Since $\mathbf{Y}$ is finite, α_U is continuous. To see that α_U preserves f, let $x \in Y$. By A1 we have $ff(x) = gf(x)$. If $f(x) \in U$, then $\alpha_U(x) = 1$ so $\alpha_U(f(x)) = 1 = f(\alpha_U(x))$. If $f(x) \notin U$, then $\alpha_U(x) = 0$ or $\alpha_U(x) = a$ so (in both cases) $\alpha_U(f(x)) = 0 = f(\alpha_U(x))$. Thus α_U preserves f, and similarly it preserves g.

The fact that α_U preserves m follows from A2, A6 and the fact that it preserves f and g. To see this, let $x, y, z \in X$ where $m(x,y) = z$. By A2 we have $f(z) = x$ and $g(z) = y$, and therefore $f(\alpha(z)) = \alpha(x)$ and $g(\alpha(z)) = \alpha(y)$. Applying A6 we obtain $m(\alpha(x), \alpha(y)) = m(f(\alpha(z)), g(\alpha(z))) = \alpha(z) = \alpha(m(x,y))$.

We now use the maps α_U to verify the conditions of the Separation Theorem.

(i) Assume $x, y \in Y$ where $x \neq y$. By A6, one of the sets $\{f(x), f(y)\}$ or $\{g(x), g(y)\}$ has two different members. Let U be a $\leq$–increasing subset of Y_0 splitting one of these sets. Then $\alpha_U(x) \neq \alpha_U(y)$.

(ii) Assume $(x,y) \notin \mathrm{dom}(m)$. First assume that x and y are both in Y_0. Then we must have $x \nleq y$. Let $U = \{z \in Y_0 \mid x \leq z\}$. Then $(\alpha_U(x), \alpha_U(y)) = (1, 0) \notin \mathrm{dom}(m)$. Now assume x and y are not both in Y_0; say $x \notin Y_0$. Then $f(x) \neq g(x)$ by A7. As $f(x) \leq g(x)$ by A6, the set $U = \{z \in Y_0 \mid g(x) \leq z\}$ contains $g(x)$ but not $f(x)$. Thus $\alpha_U(x) = a$ and so $(\alpha_U(x), \alpha_U(y)) \notin \mathrm{dom}(m)$.

It follows that $\mathbf{Y} \in \mathfrak{X}_{fgm}$. ∎

Making one last attempt at a valid conjecture, we notice that the standard examples we saw in Section 2 all arose as strong duals of algebraic quasi-varieties.

Misguided Conjecture 3.8 *A finite structure* $\underset{\sim}{\mathbf{M}}$ *is standard provided that* $R = \varnothing$, *that its quasi-equational theory* $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$ *is finitely axiomatizable and that it strongly dualises some algebraic quasi-variety.*

In fact $\underset{\sim}{\mathbf{M}}_{fgm}$ was constructed in Davey and Haviar [8] for the explicit purpose of proving and generalizing the following fact, which the authors extend to a general method of transfering a strong duality to a different generator of the quasi-variety.

Counterexample 3.9 *The partial algebra* $\underset{\sim}{\mathbf{M}}_{fgm} = \langle \{0, a, 1\}; f, g, m, \mathcal{T} \rangle$ *yields a strong duality on the quasi-variety of bounded distributive lattices relative to the generator* $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$. ([8])

Without risking any further conjectures, we conclude this section with two open problems.

Problem 3.10 *If* $\underset{\sim}{\mathbf{M}}$ *is a total algebra strongly dualising some algebra* $\underline{\mathbf{M}}$, *must it be standard?*

Problem 3.11 *If* $\underset{\sim}{\mathbf{M}}$ *is a total algebra whose quasi-equational theory has a finite basis, must it be standard?*

4. Every finite Boolean unar is standard.

Humbled by the experiences of the last section, we proceed with a healthy respect for the difficulty of the standardization problem. In this section we use Lemma 1.5 to exhibit a large class of finite structures which generate standard topological quasi-varieties. A **Boolean unar** is a Boolean structure $\mathbf{X} = \langle X; f, \mathcal{T} \rangle$ having a single unary operation f. A k–element subset $\{x_0, x_1, \ldots, x_{k-1}\}$ of X is a k–**loop** if $f(x_i) = x_{i+1(\mathrm{mod}\ k)}$ for each $i < k$. Let $L_{\mathbf{X}}$ denote the union of the loops of $\mathbf{X}$, which determines a substructure $\mathbf{L}_{\mathbf{X}}$ of $\mathbf{X}$. A **tail** of length l of $\mathbf{X}$ is an l–element subset $\{y_0, y_1, \ldots, y_{l-1}\}$ of $X \setminus L_{\mathbf{X}}$ such that $f(y_j) = y_{j+1}$ for each $j < l - 1$.

Lemma 4.1 *Let* $\mathbf{X} = \langle X; f, \mathcal{T} \rangle$ *be a Boolean unar satisfying* $f^{2m}(x) \approx f^m(x)$ *where* $m \geq 1$. *If* $x, y \in X$ *and* $x \neq y$, *then there is a finite substructure* $\mathbf{Y} \leq \mathbf{X}$ *and a morphism* $\alpha : \mathbf{X} \to \mathbf{Y}$ *separating* x *and* y.

Proof We break the proof into a series of claims.

Claim 1 *The map* $f^m : \mathbf{X} \to \mathbf{X}$ *is a retraction onto the loops of* $\mathbf{X}$.

Proof Since $\mathbf{X} \models f^{2m}(x) \approx f^m(x)$, we have $f^m(X) \subseteq L_{\mathbf{X}}$. Now let $x \in L_{\mathbf{X}}$. Then $f^k(x) = x$ for some $k \geq 1$. We have $x = (f^k)^m(x) = (f^m)^k(x) \in f^m(X)$. So $L_{\mathbf{X}} = f^m(X)$. To see that f^m is the identity on $L_{\mathbf{X}}$, let $x \in L_{\mathbf{X}}$. Then $x = f^m(y)$ for some $y \in X$, and we have $f^m(x) = f^{2m}(y) = f^m(y) = x$. ♦

Claim 2 *Let* $\mathbf{C}$ *be a component of* $\mathbf{X}$ *whose unique loop* $f^m(C)$ *has* q *elements and let* $\mathbf{D}$ *be another component of* $\mathbf{X}$ *disjoint from* $\mathbf{C}$. *Then there are clopen sets* $U_0, U_1, \dots, U_{q-1}$ *covering* C *that are disjoint from each other and from* D *such that* $f^{-1}(U_i) = U_{i+1 (\text{mod } q)}$ *for* $i < q$.

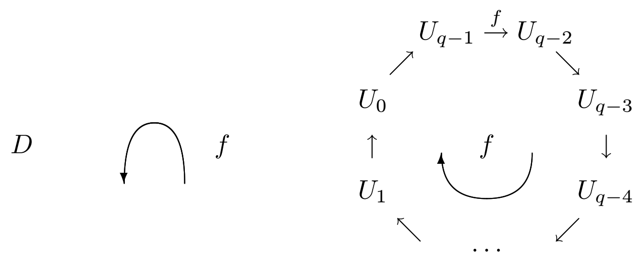

FIGURE 2. clopen cover of a component

Proof Let $x \in f^m(C)$ and let $U \subseteq X$ be a clopen set such that $x \in U$ and $((f^m(C) \setminus \{x\}) \cup D) \cap U = \varnothing$. Define

$$U' := \{ a \in X \mid \text{for all } l \geq 0,\ f^l(a) \in U \text{ if and only if } q | l \}.$$

Then $x \in U'$. The set U' is clopen since

$$U' = \cap\{ f^{-l}(U) \mid l \in \{0, \dots, 2m\} \ \& \ q|l \} \ \setminus \ \cup\{ f^{-1}(U) \mid l \in \{0, \dots, 2m\} \ \& \ q \nmid l \}.$$

Define the clopen set

$$U_i := f^{-(m+i)}(U')$$

for each $i < q$. To see that U_i is disjoint from D, suppose there were some $i < q$ such that $a \in U_i \cap D$. Then $f^{m+i}(a) \in U' \subseteq U$. But $f^{m+i}(a) \in f^m(D)$ and $f^m(D)$ is disjoint from U.

For $i < q$ we would like to show that $f^{-1}(U_i) = U_{i+1(\text{mod } q)}$. If $i < q-1$, then we have

$$f^{-1}(U_i) = f^{-1}(f^{-(m+i)}(U')) = f^{-(m+i+1)}(U') = U_{i+1}.$$

To verify that $f^{-1}(U_{q-1}) = U_0$, let $a \in f^{-1}(U_{q-1})$. Then $f(a) \in U_{q-1}$ and so $f^{m+q-1}(f(a)) = f^{m+q}(a) \in U'$. For all $l \geq 0$ we have

$$f^l(f^m(a)) \in U \Leftrightarrow f^{l+m-q}(f^{m+k}(a)) \in U \Leftrightarrow q|(l+m-q) \Leftrightarrow q|l.$$

Thus $a \in f^{-m}(U') = U_0$.

Now let $a \in U_0 = f^{-m}(U')$. We want to see that $f(a) \in U_{q-1} = f^{-(m+q-1)}(U')$. For all $l \geq 0$, we have

$$f^l(f^{m+k-1}(f(a))) \in U \Leftrightarrow f^{l+q}(f^m(a)) \in U \Leftrightarrow q|(l+q) \Leftrightarrow q|l.$$

Thus $f^{-1}(U_{q-1}) = U_0$.

If $a \in C$, then $f^{m+i}(a) = x$ for some $i < q$. Thus $a \in U_i$ and $U_0 \cup U_1 \cup \dots \cup U_{q-1}$ covers C.

It only remains to show that $U_0, U_1, \dots, U_{q-1}$ are disjoint. Suppose that $a \in U_i \cap U_j$ for some $i \leq j < q$. Then $f^{m+i}(a) \in U'$ and $f^{m+j}(a) \in U'$. Thus $j - i \geq 0$ and

$$f^{j-i}(f^{m+i}(a)) = f^{m+j}(a) \in U' \subseteq U,$$

so $q|(j-i)$ and therefore $i = j$. ⧫

Claim 3 *Let* $\mathbf{C}$ *be a component of* $\mathbf{X}$ *whose unique loop* $f^m(C)$ *has* q *elements and and whose longest tail has length* t*. Let* $x, y \in C$ *with* $x \neq y$ *and* $f^m(x) = f^m(y)$*. Then there are disjoint clopen sets* $U_0, U_1, \dots, U_{q-1}, T_0, T_1, \dots, T_d$ *with* $d < t$*, which cover* C *and separate* x *and* y*, and where*

$$f^{-1}(T_d) = \varnothing, \quad f^{-1}(T_j) = T_{j+1}, \quad f^{-1}(U_{q-1}) = T_0 \cup U_0, \quad \textit{and} \quad f^{-1}(U_i) = U_{i+1}$$

for $i < q-1$ *and* $j < d$*.*

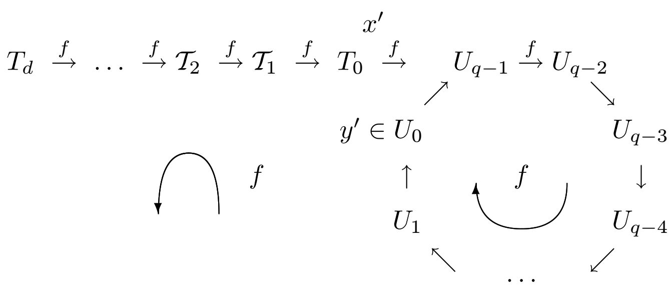

FIGURE 3. clopen cover of a component with separated tail

Proof Let $U'_0, U'_1, \dots, U'_{q-1}$ be disjoint clopen sets covering C as in Claim 2. Define

$$n := \max\{\, l \geq 0 \mid f^l(x) \neq f^l(y) \,\}$$

and $x' := f^n(x)$, $y' := f^n(y)$. In order to separate x and y we construct a morphism that separates x' and y'. We may assume that $x', y' \in U'_0$ since $f(x') = f(y')$. We can also assume that $y' \notin f^m(C)$. Define

$$d := \max\{\, l \geq 0 \mid f^{-l}(y') \neq \varnothing \,\}.$$

Then $d < t \leq m-1$.

Let $U \subseteq U'_0$ be a clopen set with $y' \in U$ and $x' \notin U$. Define $V := f^{-(d+1)}(U)$. For each $z \in U \setminus \{y'\}$, let $U_z \subseteq U$ be a clopen set with $z \in U_z$ and $y' \notin U_z$. Define $V_z := f^{-(d+1)}(U_z)$. Then $\{V_z \mid z \in U \setminus \{y'\}\}$ covers V, since $f^{-(d+1)}(y') = \varnothing$. Let $U_{\text{fin}} \subseteq U \setminus \{y'\}$ determine a finite subcover. Then

$$T_0 := U \setminus (\cup\{U_z \mid z \in U_{\text{fin}}\})$$

is a clopen subset of $U \subseteq U'_0$ containing y'.

To see that $f^{-(d+1)}(T_0) = \varnothing$, suppose that $a \in f^{-(d+1)}(T_0)$. Then $f^{d+1}(a) \in T_0 \subseteq U$. So $a \in V_z$ for some $z \in U_{\text{fin}}$. This implies that $f^{d+1}(a) \in U_z$, contradicting that fact that $T_0 \cap U_z = \varnothing$.

Now we can define $T_i := f^{-i}(T_0)$, for each $i \in \{1, 2, \dots, d\}$, and let $U_i := U'_i \setminus (T_0 \cup \dots \cup T_d)$. ⧫

Claim 4 *If* $\mathbf{Z}$ *is a clopen substructure of* $\mathbf{X}$*, then there is a morphism* $\varphi : \mathbf{Z} \to \mathbf{Z}$ *such that* $\varphi(Z)$ *is finite.*

Proof For each component $\mathbf{C}$ of $\mathbf{Z}$, let $U_0, U_1, \dots, U_{q_{\mathbf{C}}-1}$ be disjoint clopen subsets of Z which cover C, as in Claim 2, and define

$$U_{\mathbf{C}} := U_0 \cup U_1 \cup \dots \cup U_{q_{\mathbf{C}}-1}.$$

Then there is a finite set F of components of $\mathbf{Z}$ such that $\{U_{\mathbf{C}} | \mathbf{C} \in F\}$ covers Z. Let $F = \{\mathbf{C}_1, \dots, \mathbf{C}_n\}$. Then there is a morphism $\varphi : \mathbf{Z} \to \mathbf{Z}$ such that

$$\begin{aligned} U_{\mathbf{C}_1} &\mapsto f^m(C_1), \\ U_{\mathbf{C}_2} \setminus U_{\mathbf{C}_1} &\mapsto f^m(C_2), \\ &\dots, \\ U_{\mathbf{C}_n} \setminus (U_{\mathbf{C}_1} \cup \dots \cup U_{\mathbf{C}_{n-1}}) &\mapsto f^m(C_n), \end{aligned}$$

and the range of φ is clearly finite. ♦

We can now prove Lemma 4.1. Let $x, y \in X$ where $x \neq y$. Let $\mathbf{C}_x$ and $\mathbf{C}_y$ be the components of $\mathbf{X}$ containing x and y.

`Case (1):` $\mathbf{C}_x \neq \mathbf{C}_y$. By Claim 2 there are clopen sets $U_0, U_1, \dots, U_{q-1}$ of X such that $Z := U_0 \cup U_1 \cup \dots \cup U_{q-1}$ and $Z' := X \setminus Z$ form a clopen substructures $\mathbf{Z}$ and $\mathbf{Z}'$ of $\mathbf{X}$ with $C_x \subseteq Z$ and $C_y \subseteq Z'$. By Claim 3 there are morphisms $\varphi_x : \mathbf{Z} \to \mathbf{Z}$ and $\varphi_y : \mathbf{Z}' \to \mathbf{Z}'$ with finite ranges. Then $\varphi = \varphi_x \cup \varphi_y : \mathbf{X} \to \mathbf{X}$ separates x and y and has finite range.

`Case (2):` $\mathbf{C}_x = \mathbf{C}_y$ and $f^m(x) \neq f^m(y)$. Choose clopen sets $U_0, U_1, \dots, U_{q-1}$ covering C_x as in Claim 2. These sets separate $f^m(x)$ and $f^m(y)$, and therefore also separate x and y. We can now use Claim 4 to define a morphism $\varphi : \mathbf{X} \to \mathbf{X}$ in such a way that $U_0 \cup U_1 \cup \dots \cup U_{q-1}$ maps onto $f^m(C_x)$, that $\varphi(X)$ is finite and that $\varphi(x) \neq \varphi(y)$.

`Case (3):` $\mathbf{C}_x = \mathbf{C}_y$ and $f^m(x) = f^m(y)$. We do here as in Case 2 but use Claim 3 instead of Claim 2. ∎

Theorem 4.2 *Every finite Boolean unar is standard.*

Proof Consider a finite Boolean unar $\underset{\sim}{\mathbf{M}} = \langle M; f, \mathcal{T} \rangle$. Let n be the least common multiple of the sizes of the loops of $\underset{\sim}{\mathbf{M}}$, let t be the length of the longest tail in $\underset{\sim}{\mathbf{M}}$ and let $m = nt$. Notice that f^m maps $\underset{\sim}{\mathbf{M}}$ into $L_{\underset{\sim}{\mathbf{M}}}$ and fixes each element of $L_{\underset{\sim}{\mathbf{M}}}$. Thus $\underset{\sim}{\mathbf{M}}$ satisfies $f^{2m}(x) \approx f^m(x)$.

Now let $\mathbf{X} \in \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}}))$. Then $\mathbf{X}$ also satisfies $f^{2m}(x) \approx f^m(x)$. Choose $x, y \in X$ where $x \neq y$. By Lemma 4.1 there is a finite $\mathbf{Y} \leq \mathbf{X}$ and a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $\alpha(x) \neq \alpha(y)$. By Lemma 1.1, $\mathbf{Y} \in \mathfrak{X}$, and consequently there is a morphism $\beta : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ such that $(\beta \circ \alpha)(x) \neq (\beta \circ \alpha)(y)$. It follows from the Separation Theorem that $\mathbf{X} \in \mathfrak{X}$. ∎

5. Every finite Boolean unar is finitely axiomatizable.

Knowing in advance that the topological quasi-variety generated by a finite Boolean unar $\underset{\sim}{\mathbf{M}} = \langle M; f, \mathcal{T} \rangle$ is standard will now vastly simplify the axiomatization process. Applying Corollary 1.4 we will show that a particular finite set Σ axiomatizes $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$. Working only with finite structures, we are now freed of the obligation to continue patching together clopen sets.

Again we let n be the least common multiple of the sizes of the loops of $\underset{\sim}{\mathbf{M}}$, let t be the length of the longest tail in $\underset{\sim}{\mathbf{M}}$ and let $m = nt$. Our proof that $\underset{\sim}{\mathbf{M}}$ is standard used only the fact that $\underset{\sim}{\mathbf{M}}$ satisfies the equation $f^{2m}(x) \approx f^m(x)$. In general this single equation does not axiomatize $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$. For example, if $\underset{\sim}{\mathbf{M}}$ consists of a 6-loop and a 10-loop, then $\underset{\sim}{\mathbf{M}} \models f^{60}(x) \approx f^{30}(x) \approx x$. This equation is also satisfied by a 15-loop, but the loops of a power of $\underset{\sim}{\mathbf{M}}$ all have size 6, 10 or 30; not 15.

In order to exhibit a complete set of axioms for $\mathrm{Th}_{qa}(\underset{\sim}{\mathbf{M}})$, let Q be the set of positive integers that are the least common multiple (lcm) of the sizes of some set of loops of $\underset{\sim}{\mathbf{M}}$. Thus

$$Q \subseteq \{1, 2, \ldots, n\}.$$

Notice that the lcm of a subset of Q is again in Q. For $a \in \underset{\sim}{\mathbf{M}}$, the subalgebra of $\underset{\sim}{\mathbf{M}}$ generated by a is the disjoint union of a (possibly empty) tail, T_a, and a loop, L_a.

If $a \in M$, then $f^t(a) \in L_a$ and $|L_a|$ divides n. Thus $f^t(a)$ is fixed by f^n, that is, $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^{t+n}(x) \approx f^t(x). \tag{A}$$

For each $q \in Q$, let $t(q)$ be the length of the longest tail attached to a loop of $\underset{\sim}{\mathbf{M}}$ whose size divides q. Suppose that $a \in M$ and $f^{t+q}(a) = f^t(a)$. Since $f^t(a) \in L_a$, this tells us that $|L_a|$ divides q. Thus $|T_a| \leq t(q)$, and therefore $f^{t(q)+q}(a) = f^{t(q)}(a)$. This tells us that $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^{t+q}(x) \approx f^t(x) \implies f^{t(q)+q}(x) \approx f^{t(q)}(x) \tag{B$_q$}$$

for each $q \in Q$.

Now suppose that $a \in M$ and $f^r(a) = a$, and let $q = |L_a|$. Then $q \in Q$ and $q|r$. Thus $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^r(x) \not\approx x \tag{C$_r$}$$

for each positive integer r which is not a multiple of any member of Q.

On the other hand, assume that r is a multiple of some member of Q and let $s \in Q$ be the lcm of the members of Q which divide r. Then $s|r$. Again suppose that $a \in M$ and $f^r(a) = a$, and let $q = |L_a|$. Then $q|r$ so $q|s$, and therefore $f^s(a) = a$. Thus $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^r(x) \approx x \implies f^s(x) \approx x \tag{D$_r$}$$

where some member of Q divides r, and s is the lcm of all members of Q that divide r.

Finally, it may happen that $\underset{\sim}{\mathbf{M}}$ contains exactly one 1-element loop, which is expressed by the implication

$$f(x) \approx x \;\&\; f(y) \approx y \implies x \approx y. \tag{E}$$

We now define Σ to consist of

(A),
(B$_q$) for each $q \in Q$,
(C$_r$) for each $r \leq n$ which is not a multiple of any member of Q,
(D$_r$) for each $r \leq n$ which is a multiple of some member of Q, and
(E) just in case $\underset{\sim}{\mathbf{M}}$ has exactly one 1-element loop.

Theorem 5.1 *If $\underset{\sim}{\mathbf{M}}$ is a finite Boolean unar, then $\Sigma_{\underset{\sim}{\mathbf{M}}}$ is a finite axiomatization for $\mathfrak{X} = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$.*

Proof We apply Corollary 1.4. Clearly every model of (A) is locally finite. Let $\mathbf{X} = \langle X; f, \mathcal{T} \rangle$ be a finite model of Σ.

We first observe that the size of each loop L of $\mathbf{X}$ is in Q. To see this, let $|L| = r$. Then $r \leq n$ by (A). If $x \in L$, then $f^r(x) = x$ so $\mathbf{X}$ does not satisfy (C_r). Thus $(\mathrm{C}_r) \notin \Sigma$ and we conclude that r is a multiple of some member of Q. Let s be the lcm of all members of Q that divide r. Then $f^s(x) = x$ by (D_r). Thus $r|s$. But clearly $s|r$, so $r = s \in Q$.

Next we observe that if L is a loop of $\mathbf{X}$, then $\mathbf{L} = \langle L; f, \mathcal{T} \rangle$ can be embedded into a finite power of $\underset{\sim}{\mathbf{M}}$. To prove this, let $|L| = r \in Q$. Then $r = \mathrm{lcm}\{l_0, l_1, \dots, l_{k-1}\}$ where $g_i \in M$ generates an l_i–loop for each $i < k$. Now $g := (g_0, g_1, \dots, g_{k-1})$ generates an r–loop of $\underset{\sim}{\mathbf{M}}^k$ isomorphic to $\mathbf{L}$.

We will now prove that $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ by showing that there is at least one morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ and then verifying condition (i) of the Separation Theorem 0.1. To produce one morphism, we have $f^t : \mathbf{X} \to \mathbf{L}_\mathbf{X}$ by (A). Each loop of $\mathbf{X}$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$ which, in turn, can be mapped into $\underset{\sim}{\mathbf{M}}$. Using the fact that $\mathbf{L}_\mathbf{X}$ is the coproduct of the loops of $\mathbf{X}$ gives us one morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$.

Now consider $x, y \in X$ where $x \neq y$.

`Case (1):` x and y are in different connected components, $\mathbf{C}_x$ and $\mathbf{C}_y$, of $\mathbf{X}$. By (A) the morphism f^t maps $\mathbf{C}_x$ and $\mathbf{C}_y$, respectively, into the loops $\mathbf{L}_x$ and $\mathbf{L}_y$ of $\mathbf{X}$.

Suppose that one of these, say $\mathbf{L}_y$, is not a 1–loop. Then $f^{t+1}(y) \neq f^t(y)$. Since $\mathbf{L}_x$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$, there is a morphism $\alpha : \mathbf{L}_x \to \underset{\sim}{\mathbf{M}}$. Since $\mathbf{L}_y$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$, there is a morphism $\beta : \mathbf{L}_y \to \underset{\sim}{\mathbf{M}}$ where $\beta(f^{t+1}(y)) \neq \beta(f^t(y))$. Let $\gamma := (\beta \circ f^{t+1}) \cup (\alpha \circ f^t)$ and $\delta := (\beta \circ f^t) \cup (\alpha \circ f^t)$. Then γ and δ are both morphisms from $\mathbf{L}_x \cup \mathbf{L}_y$ into $\underset{\sim}{\mathbf{M}}$, and one of them separates x and y. This one can then be extended to the other components of $\mathbf{X}$.

Now suppose that both $\mathbf{L}_x$ and $\mathbf{L}_y$ are 1–loops. Since they can both be embedded into $\underset{\sim}{\mathbf{M}}$, we conclude that $\underset{\sim}{\mathbf{M}}$ contains at least one 1–loop. Since $\mathbf{X}$ has two 1–loops, it does not satisfy (E). Therefore (E) is not in Σ, that is, $\underset{\sim}{\mathbf{M}}$ has more than one 1–loop. We can then map $\mathbf{L}_x$ and $\mathbf{L}_y$ to separate 1–loops of $\underset{\sim}{\mathbf{M}}$, and extend these to a single map of $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ separating x and y.

`Case (2):` x and y are in the same connected component $\mathbf{C}$ of $\mathbf{X}$ but $f^t(x) \neq f^t(y)$. Then $f^t : \mathbf{C} \to \mathbf{L}_x = \mathbf{L}_y$ separates x and y. Embedding $\mathbf{L}_x$ into a power of $\underset{\sim}{\mathbf{M}}$ and mapping then into $\underset{\sim}{\mathbf{M}}$ we can separate x and y with a morphism that extends to a morphism $\alpha : \mathbf{L}_\mathbf{X} \to \underset{\sim}{\mathbf{M}}$. Now $\alpha \circ f^t : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ separates x and y.

`Case (3):` x and y are in the same connected component $\mathbf{C}$ of $\mathbf{X}$ and $f^t(x) = f^t(y)$. Since $x \neq y$, it cannot be that both x and y are in $L_\mathbf{X}$. Consequently it cannot be that there are positive integers i and j where $f^i(x) = y$ and $f^j(y) = x$. Assume that there is no i such that $f^i(x) = y$. In particular, $y \notin L_\mathbf{X}$.

Let $L_x = L_y = \{x_0, x_1, \dots, x_{q-1}\}$ be distinct, where $f(x_i) = x_{i-1 (\mathrm{mod}\ q)}$ for $i < q$. Since $y \notin L_\mathbf{X}$, there is an integer s, where $0 < s < t$, such that $f^{-s}(y) = \varnothing$ but $f^{-(s-1)} \neq \varnothing$. This gives us a partition

$$A := f^{-(s-1)}(y) \cup \cdots \cup f^{-3}(y) \cup f^{-2}(y) \cup f^{-1}(y) \cup \{y\}$$

of the set A of ancestors of y, where $f(f^{-j}(y)) \subseteq f^{-(j-1)}(y)$ for $1 \leq j < s$. Since $f^t : \mathbf{C} \to \mathbf{L}_x$, the kernel of f^t is a congruence on $\mathbf{C}$ given by the partition

$$C = f^{-t}(x_0) \cup f^{-t}(x_1) \cup \cdots \cup f^{-t}(x_{q-1}).$$

where

$$f(f^{-t}(x_i)) \subseteq f^{-t}(x_{i-1(\text{mod } q)})$$

for all $i < q$. Let $C_i := f^{-t}(x_i) \setminus A$ for each $i < q$. Since $f^t(x) = f^t(y)$, neither $f(x)$ nor $f(y)$ is in A, so we may assume that $f(x), f(y) \in C_{q-1}$. Then $x \in C_0$. This gives us a refined partition of C which forms a congruence shown in Figure 4. Using this congruence we can separate x and y by a morphism into $\underset{\sim}{\mathbf{M}}$ provided that we can find within $\underset{\sim}{\mathbf{M}}$ an element b where $|T_b| \geq s$ and $|L_b|$ divides q.

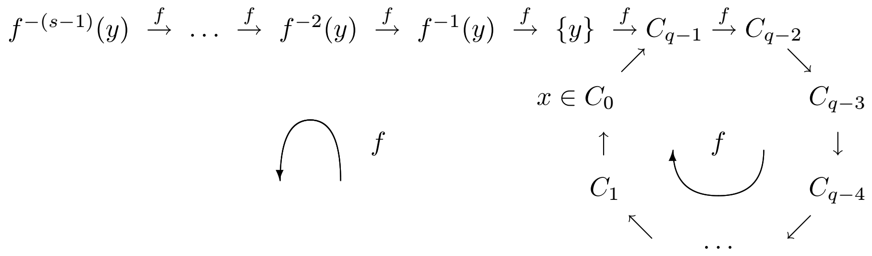

FIGURE 4. a congruence on the component $\mathbf{C}$

Let $z \in f^{-(s-1)}(y)$. Then $|T_z| = s$, and $L_z = L_y$ has size q. We now apply (B_q), which says that if an element has a loop whose size divides q, then its tail has length at most $t(q)$. Thus $t(q) \geq s$. From the definition of $t(q)$, there is an element $b \in M$ such that $|T_b| = t(q) \geq s$ and $|L_b|$ divides q. This guarantees that x and y can be separated by a morphism from $\mathbf{C}$ into $\underset{\sim}{\mathbf{M}}$, which extends to all of $\mathbf{X}$ since each other component of $\mathbf{X}$ can also be mapped into $\underset{\sim}{\mathbf{M}}$. ∎

REFERENCES

1. H. Andréka, P. Burmeister and I. Németi, Quasivarieties of partial algebras. A unifying approach towards a two-valued model theory for partial algebras, *Studia Sci. Math. Hungar.* **16** (1983), No. 3-4, 325–372.
2. I.P. Bestsennyi, Quasiidentities of finite unary algebras, *Algebra i Logika* **28**, No. 5 (1989), 493-512. (English translation: UDC 512.579, Plenum Publishing Co. 1990)
3. D.M. Clark and B.A. Davey, When is a natural duality 'good'?, *Algebra Universalis* **35** (1996), 265–295.
4. D.M. Clark and B.A. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge (1998).
5. D.M. Clark and P.H. Krauss, Topological quasivarieties, *Acta Sci. Math.*(Szeged) **47** (1984), 3–39.
6. W.H. Cornish and P.R. Fowler, Coproducts of De Morgan algebras, *Bull. Austral. Math. Soc.* **16** (1977), 1–13.
7. B.A. Davey, Dualities for Stone algebras, double Stone algebras and relative Stone algebras, *Coll. Math.* **46** (1982), 1–14.
8. B.A. Davey and M. Haviar, A schizophrenic operation which aids the efficient transfer of strong dualities, *Houston J. of Math.* **26** (2000), 215–222.

9. B.A. Davey, M. Haviar and H.A. Priestley, Endoprimal distributive lattices are endodualisable, *Algebra Universalis* **34** (1995), 444–453.
10. B.A. Davey and H.A. Priestley, Optimal natural dualities, *Trans. Amer. Math. Soc.* **338** (1993), 655–677.
11. B.A. Davey and H.A. Priestley, Optimal natural dualities II: general theory, *Trans. Amer. Math. Soc.* **348** (1996), 3673–3711.
12. B.A. Davey and H.A. Priestley, Optimal natural dualities for varieties of Heyting algebras, *Studia Logica* **56** (1996), 67–96.
13. B.A. Davey and H. Werner, Dualities and equivalences for varieties of algebras, *Contributions to Lattice Theory*(Szeged, 1980), North-Holland, Amsterdam (1983), 101–275.
14. K.H. Hofmann, M. Mislove and A. Stralka, *The Pontryagin duality of compact* 0*-dimensional semi-lattices and its applications*, Lecture Notes in Mathematics **396**, Springer-Verlag, Berlin, New York (1974).
15. T.K. Hu, On the topological duality for primal algebra theory, *Algebra Universalis* **1** (1971), 152–154.
16. J. Isbell, Median algebras, *Trans. Amer. Math. Soc.* **260** (1980), 319–362.
17. I.A. Mal'cev, On the general theory of algebraic systems (Russian) *Mat. Sb.* **35** (1954), 3–20.
18. H.A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.* **2** (1972), 186–190.
19. M.J.A.D.G. Saramago, *A Study of Natural Dualities, Including an Analysis of the Structure of Failsets*, D.Phil. Thesis, Universidade de Lisboa (1998).
20. M.H. Stone, The theory of representations for Boolean algebras, *Trans. Amer. Math. Soc.* **4** (1936), 37–111.
21. A. Stralka, A partially ordered space which is not a Priestley space, *Semigroup Forum* **20** (1980), 293–297.
22. C. Wegener, *Natural Dualities for Varieties Generated by Lattice-Structured Algebras*, D.Phil. Thesis, Oxford University (1999).
23. H. Werner, A duality for weakly associative lattices, *Finite algebra and multiple-valued logic*, Szeged (1979), *Colloq. Math. Soc. János Bolyai* **28**, North-Holland, Amsterdam, New York (1981), 781–808.

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