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Trojicové konštrukcie MS-algebier

(ŠVOČ)

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On triple construction of MS-algebras

(Student Competition)

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CONTENTS

1. INTRODUCTION

In 1980 T. S. Blyth and J. C. Varlet presented the first triple construction of MS-algebras from the subvariety K_2 by means of Kleene algebras and distributive lattices [5]. In [6] this construction was improved via the language of quadruples. It was independentely done by T. Katriňák and K. Mikula (in an unpublished paper) who then compared both approaches in [12].

Later, M. Haviar [8] proved that there exists a one-to-one correspondence between the class of locally bounded K_2-algebras and the class of decomposable K_2-quadruples. In his work he assumed that the filter $L^\vee$ of an MS-algebra L was principal which allowed him to simplify the previous constructions and to work with pairs of elements only. A year later in [9] he presented a similar triple construction of principal p-algebras.

In Section 3 of this thesis we present a simple triple construction of principal MS-algebras analogous to that of [9] and we show that there is a one-to-one correspondence between principal MS-algebras and so-called principal MS-triples.

We also introduce a class of so-called decomposable MS-algebras containing the class of principal MS-algebras and we present a triple construction of decomposable MS-algebras generalising that in Section 3. It is a modification of the quadruple constructions by T. S. Blyth and J. C. Varlet ([5], [6]) and T. Katriňák and K. Mikula ([12]).

Firstly, we use de Morgan algebras instead of Kleene algebras in our triples and secondly, the filter chosen for our construction is different. Instead of the filter $L^\vee$ used in the constructions in [5], [6],[12] and [8], in our constructions in Sections 3 and 4 we consider the set $D(L)$ of dense elements of an MS-algebra L. As $D(L)$ is a filter for any MS-algebra L we do not need a quadruple to construct an MS-algebra. It is sufficient to use the triple construction, because we do not need to use the modal operator used in the constructions by T. S. Blyth and J. C. Varlet ([5], [6]) and T. Katriňák and K. Mikula ([12]) or the congruence used by M. Haviar ([8]).

2. PRELIMINARIES

An *MS-algebra* is an algebra $(L; \vee, \wedge, ^0, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and ^{0} is a unary operation such that for all $x, y \in L$

$(MS1)$ $x \leq x^{00}$;
$(MS2)$ $(x \wedge y)^0 = x^0 \vee y^0$;
$(MS3)$ $1^0 = 0$.

The class of all MS-algebras is equational. Algebras from the subvariety M (*de Morgan algebras*) satisfy the additional identity

(4) $x = x^{00}$.

A de Morgan algebra satisfying the identity

(5) $(x \wedge x^0) \vee y \vee y^0 = y \vee y^0$

is called a *Kleene algebra*.

Let L be an MS-algebra. Then

(i) $L^{00} = \{x \in L \mid x = x^{00}\}$ is a de Morgan algebra;
(ii) $D(L) = \{x \in L \mid x^0 = 0\}$ is a filter (of dense elements) of L.

The following definition mimics the one in [9]:

Definition 2.1. *An MS-algebra $(L; \vee, \wedge, ^0, 0, 1)$ is called a* **principal MS-algebra**, *if it satisfies the following conditions:*

(i) *The filter $D(L)$ is principal, i.e. there exists an element $d_L \in L$ such that $D(L) = [d_L)$;*
(ii) $x = x^{00} \wedge (x \vee d_L)$ *for any $x \in L$.*

Now we introduce a new concept of a decomposable MS-algebra generalising the concept of a principal MS-algebra.

Definition 2.2. *An MS-algebra $(L; \vee, \wedge, ^0, 0, 1)$ will be called a* **decomposable MS-algebra** *if*

(i) *for every $x \in L$ there exists $d \in D(L)$ such that $x = x^{00} \wedge d$;*
(ii) *$x^{00} = y^{00}$ and $[x) \cap D(L) = [y) \cap D(L)$ imply $x = y$ for all $x, y \in L$.*

Let L be a principal MS-algebra with $D(L) = [d_L)$ and for $x \in L$ let $d := x \vee d_L$. Then $d \in [d_L)$ and the condition (ii) of Definition 2.1 gives us $x = x^{00} \wedge d$. Thus the condition (i) of Definition 2.2 is satisfied.

Now let $x^{00} = y^{00}$ and $[x) \cap D(L) = [y) \cap D(L)$ for $x, y \in L$. Then

$$\begin{aligned} [x \vee d_L) &= [x) \cap [d_L) \\ &= [x) \cap D(L) \\ &= [y) \cap D(L) \\ &= [y) \cap [d_L) \\ &= [y \vee d_L)\,, \end{aligned}$$

consequently

$$\begin{aligned} x &= x^{00} \wedge d \\ &= x^{00} \wedge (x \vee d) \\ &= y^{00} \wedge (y \vee d) \\ &= y^{00} \wedge d \\ &= y. \end{aligned}$$

This shows that also the condition (ii) of Definition 2.2 is satisfied in any principal MS-algebra.

3. PRINCIPAL MS-ALGEBRAS

In this section we give a construction of principal MS-algebras which works with pairs of elements only and mimics the construction of principal p-algebras from [9].

Definition 3.1. *An (abstract)* **principal MS-triple** *is (M, D, φ), where*

(i) *M is a de Morgan algebra;*
(ii) *D is a bounded distributive lattice;*
(iii) *φ is a $(0,1)$-homomorphism from M into D.*

Theorem 3.2. *Let (M, D, φ) be a principal MS-triple. Then*

$$L = \{(x, y) \mid x \in M, y \in D, y \leq \varphi(x)\}$$

is a principal MS-algebra, if we define

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2)$$

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2)$$

$$(x, y)^0 = (x^0, \varphi(x^0))$$

$$1_L = (1, 1)$$

$$0_L = (0,0).$$

Moreover, $L^{00} \cong M$ *and* $D(L) \cong D$.

Proof. One can easily prove that L is a sublattice of $M \times D$. Obviously, $0_D = \varphi(0_M)$ and $1_D = \varphi(1_M)$. Hence, L is a bounded distributive lattice. Clearly,

$$(x,y) \wedge (x,y)^{00} = (x \wedge x^{00}, y \wedge \varphi(x^{00})) = (x,y),$$

so the identity (1) hold in L. We can verify the identities (2) and (3) similarly.

Now,

$$\begin{aligned}
D(L) &= \{(x,y) \in L \mid (x,y)^0 = (0_M, 0_D)\} \\
&= \{(x,y) \in L \mid (x^0, \varphi(x^0)) \\
&= (0_M, 0_D)\} \\
&= \{(1_M, y) \mid y \in D\} \\
&\cong D.
\end{aligned}$$

Evidently, an element $d_L = (1_M, 0_D)$ is the smallest dense element of L and the filter $D(L)$ is principal.

Also, for any $(x,y) \in L$,

$$\begin{aligned}
(x,y)^{00} \wedge ((x,y) \vee (1_M, 0_D)) &= (x^{00}, \varphi(x^{00})) \wedge (x \vee 1_M, y \vee 0_D) \\
&= (x, \varphi(x)) \wedge (1_M, y) \\
&= (x,y).
\end{aligned}$$

Hence L is a principal MS-algebra.

It remains to prove that $L^{00} \cong M$. We have

$$\begin{aligned}
L^{00} &= \{(x,y) \in L \mid (x,y)^{00} = (x,y)\} \\
&= \{(x,y) \in L \mid (x^{00}, \varphi(x^{00})) = (x,y)\} \\
&= \{(x,y) \mid x \in M, y \in D, y = \varphi(x)\} \\
&= \{(x, \varphi(x)) \mid x \in M\},
\end{aligned}$$

which is obviously isomorphic to M. The proof is complete. □

We shall say that the principal MS-algebra L from Theorem 3.2 is *associated* with the principal MS-triple (M, D, φ) and the construction of L described in Theorem 3.2 will be called a *principal MS-construction.*

We illustrate the principal MS-construction on the following example.

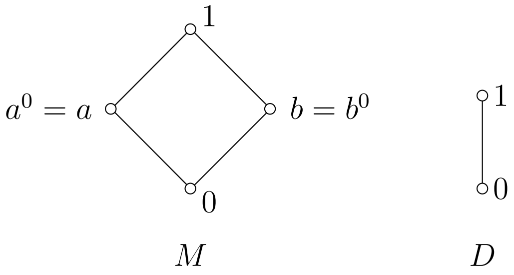

Figure 1

Example 3.3. Let M be the four-element subdirectly irreducible de Morgan algebra and let D be the two-element lattice (see Figure 1).

Define a homomorphism $\varphi : M \to D$ by the rule

$$\varphi(0) = \varphi(a) = 0,$$

$$\varphi(b) = \varphi(1) = 1.$$

Then (M, D, φ) is a principal MS-triple and by the principal MS-construction we obtain a principal MS-algebra L such that

$$L = \{(0,0), (a,0), (b,0), (b,1), (1,0), (1,1)\}$$

and

$$(0,0)^0 = (1,1), (a,0)^0 = (a,0), (b,0)^0 = (b,1)^0 = (b,1), (1,0)^0 = (1,1)^0 = (0,0).$$

The algebra L is represented in Figure 2. The shaded elements form a de Morgan algebra L^{00} which is obviously isomorphic to M. One can also observe that the filter $D(L)$ is isomorphic to the given lattice D. Moreover, the homomorphism $\varphi(L) : L^{00} \to D(L)$ defined by $\varphi(L)(x,y) = (x,y) \vee (1,0)$ is a $(0,1)$-homomorphism. Hence the triple $(L^{00}, D(L), \varphi(L))$ is a principal MS-triple.

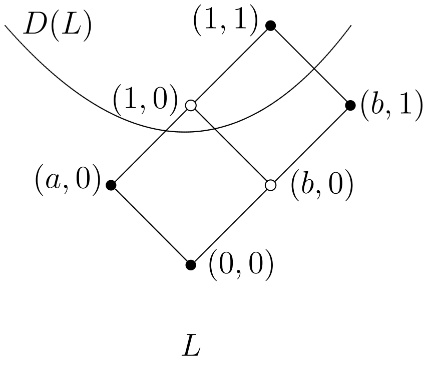

Figure 2

■

Let L be a principal MS-algebra and let d_L be the smallest dense element of L. Define a mapping $\varphi(L) : L^{00} \to D(L)$ by $\varphi(L)(a) = a \vee d_L$. It is obvious that $\varphi(L)$ is a $(0,1)$-homomorphism.

We say that $(L^{00}, D(L), \varphi(L)$ is a principal MS-triple *associated* with L.

The following theorem states that every principal MS-algebra can be obtained by the principal MS-construction.

Theorem 3.4. *Let L be a principal MS-algebra. Let $(L^{00}, D(L), \varphi(L))$ be the principal MS-triple associated with L. Then the principal MS-algebra L_1 associated with $(L^{00}, D(L), \varphi(L))$ is isomorphic to L.*

Proof. Let $D(L) = [d_L)$. We have to prove that the mapping $f : L \to L_1$ defined by

$$f(a) = (a^{00}, a \vee d_L)$$

is the desired isomorphism. It is obvious that $f(a) \in L'$, as

$$a \vee d_L \leq \varphi(a^{00}) = a^{00} \vee d_L.$$

It is easy to prove that f is a lattice homomorphism and that $f(0) = (0, d_L)$ and $f(1) = (1, 1)$. Moreover, we have

$$\begin{aligned} f(a^0) &= (a^{000}, a^0 \vee d_L) \\ &= (a^0, \varphi(a^0)) \\ &= (f(a))^0, \end{aligned}$$

so f is a homomorphism of MS-algebras.

Now we will prove the injectivity. Assume that $f(a_1) = f(a_2)$. Then we have $a_1^{00} = a_2^{00}$ and $a_1 \vee d_L = a_2 \vee d_L$ and we immediately obtain

$$a_1^{00} \wedge (a_1 \vee d_L) = a_2^{00} \wedge (a_2 \vee d_L).$$

Thus $a_1 = a_2$.

To prove the surjectivity of f, let $(x, y) \in L_1$. Set $a = x \wedge y$. Using the facts that $x \in L^{00}$, $y \in D(L)$ and $y \leq \varphi(x)$, we get

$$\begin{aligned} f(a) &= ((x \wedge y)^{00}, (x \wedge y) \vee d_L) \\ &= (x^{00} \wedge y^{00}, (x \vee d_L) \wedge (y \vee d_L)) \\ &= (x \wedge 1_L, (x \vee d_L) \wedge y) \\ &= (x, \varphi(x) \wedge y) \\ &= (x, y). \end{aligned}$$

The proof is complete. □

Definition 3.5. *An isomorphism of principal MS-triples (M, D, φ) and (M_1, D_1, φ_1) is a pair (f, g) where f is an isomorphism of M and M_1, g is an isomorphism of D and D_1 and the diagram*

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, "f"'] & M_1 \arrow[d, "g"] \\ D \arrow[r, "\varphi_1"] & D_1 \end{tikzcd}$$

is commutative.

Theorem 3.6. *Two principal MS-algebras are isomorphic if and only if their associated principal MS-triples are isomorphic.*

Proof. Let $h : L_1 \to L_2$ be an isomorphism of MS-algebras. Then the restrictions $h\,|_{L^{00}}$ and $h\,|D(L)$ are the required isomorphisms.

Conversely, let (M_1, D_1, φ_1) and (M_2, D_2, φ_2) be the principal MS-triples associated to principal MS-algebras L_1 and L_2 and let

$$(f, g) : (M_1, D_1, \varphi_1) \to (M_2, D_2, \varphi_2)$$

be an isomorphism of principal MS-triples. Let us denote by L_1' and L_2' the principal MS-algebras constructed from the principal MS-triples (M_1, D_1, φ_1) and (M_2, D_2, φ_2), respectively, by the principal MS-construction. Consider the mapping $h : L_1' \to L_2'$ defined by the rule $h((a, x)) = (f(a), g(x))$. It is clear that h is a lattice isomorphism. Moreover, we have

$$\begin{aligned} h((a,x)^0) &= h(a^0, \varphi(a^0)) \\ &= (f(a^0), g(\varphi_1(a^0))) \\ &= (f(a^0), \varphi_2(f(a^0))) \\ &= ((f(a))^0, \varphi_2(f(a))^0) \\ &= (f(a), g(x))^0 \\ &= (h(a,x))^0. \end{aligned}$$

Hence h is an isomorphism of MS-algebras. □

The next theorem together with the previous two theorems show that there is a one-to-one correspondence between the principal MS-algebras and principal MS-triples.

Theorem 3.7. *Let* (M, D, φ) *be a principal* MS*-triple and let* L *be its associated principal* MS*-algebra. Then*

$$(L^{00}, D(L), \varphi(L)) \cong (M, D, \varphi).$$

Proof. By Theorem 3.2 the mappings $f : L^{00} \to M$ and $g : D(L) \to D$ such that $f(a, \varphi(a)) = a$ and $g(1_M, x) = x$, are isomorphisms. It remains to prove that the diagram

$$\begin{tikzcd}
L^{00} \arrow[r, "\varphi(L)"] \arrow[d, "f"'] & D(L) \arrow[d, "g"] \\
M \arrow[r, "\varphi"] & D
\end{tikzcd}$$

is commutative. Let $u \in L^{00}$. Then $u = (a, \varphi(a))$ for some $a \in M$ and we have

$$\begin{aligned}
g(\varphi(L)(u)) &= g((a, \varphi(a)) \vee (1_M, 0_D)) \\
&= g(a \vee 1_M, \varphi(a) \vee 0_D) \\
&= g(1_M, \varphi(a)) \\
&= \varphi(a) \\
&= \varphi(f(a, \varphi(a))).
\end{aligned}$$

The proof is complete. □

Hence, here the situation is different to [8], where it was possible to construct an MS-algebra from the subvariety K_2 from two non-isomorphic K_2-quadruples.

Example 3.8. Let K be the three-element subdirectly irreducible Kleene algebra and let D be the two-element lattice. Define two homomorphisms $\varphi_1, \varphi_2 : K \to D$, by the rules

$$\varphi_1(0) = \varphi_1(a) = 0, \varphi_1(1) = 1$$

and

$$\varphi_2(0) = 0, \varphi_2(a) = \varphi_2(1) = 1$$

(see Figure 3).

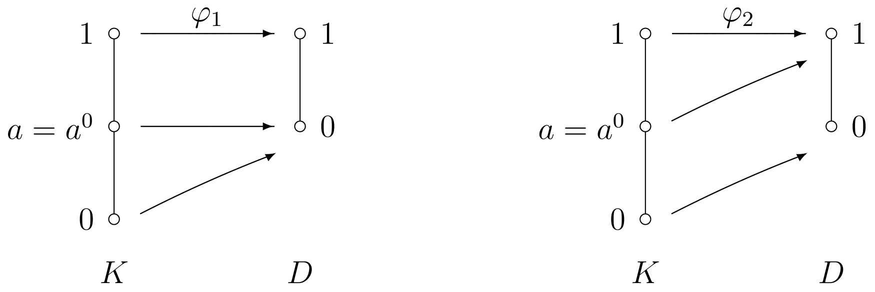

Figure 3

By the principal constructions, from the principal MS-triples (K, D, φ_1) and (K, D, φ_2) we obtain the non-isomorphic principal MS-algebras L_1 resp. L_2 depicted in Figure 4.

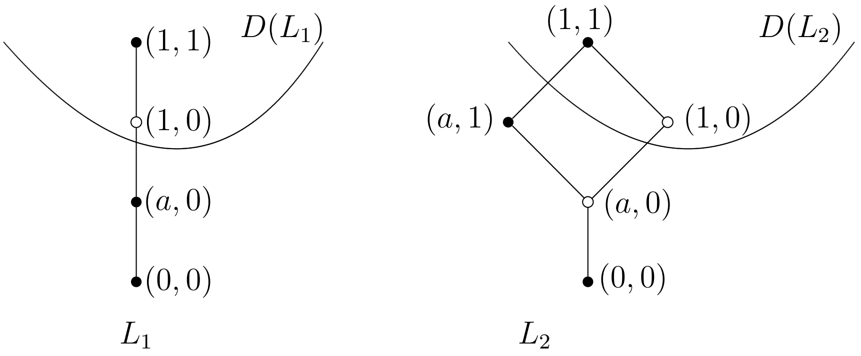

Figure 4

One can easily observe that $L_1^{00} \cong L_2^{00}$ (Kleene algebras L_1^{00}, L_2^{00} are shaded) and $D(L_1) \cong D(L_2)$, but $\varphi(L_1) \neq \varphi(L_2)$. So taking two different $(0,1)$-homomorphisms between a de Morgan algebra and a bounded distributive lattice leads to obtaining two non-isomorphic principal MS-algebras by the principal MS-construction.

■

4. Decomposable MS-algebras

In this section we present a construction of decomposable MS-algebras. As the class of decomposable MS-algebras includes the class of principal MS-algebras, the construction given in this section generalises the one given in Theorem 3.2.

Our construction is similar to those by T. S. Blyth and J. C. Varlet ([5], [6]) and T. Katriňák and K. Mikula ([12]). However, working with the filter $D(L)$ instead of $L^{\vee}$ enables us to use the triple construction only. Also we use de Morgan algebras instead of Kleene algebras in our triples. Consequentely, we construct the decomposable MS-algebras not only from the subvariety K_2.

For a decomposable MS-algebra L we will use the notation $F(D(L))$ for the lattice of all filters of $D(L)$ and the notation $F_d(D(L))$ for the dual lattice of the lattice $F(D(L))$.

We consider the mapping $\varphi(L) : L^{00} \to F(D(L))$ defined by

$$\varphi(L)(a) = \{x \in D(L) \mid x \geq x^0\} = \left[a^0\right) \cap D(L), \quad a \in L^{00}.$$

For a decomposable MS-algebra L the mapping $\varphi(L)$ defined above is a $(0,1)$-homomorphism from L^{00} into $F(D(L))$ and, moreover, $\varphi(L)(a) \cap [y)$ is a principal filter of $D(L)$ for every $a \in L^{00}$ and for every $y \in D(L)$.

Definition 4.1. *A* **decomposable** MS**-triple** *is* (M, D, φ), *where*

(i) M *is a de Morgan algebra;*
(ii) D *is a distributive lattice with* 1*;*
(iii) φ *is a* $(0,1)$*-homomorphism from* M *into* $F(D)$ *such that for every element* $a \in M$ *and for every* $y \in D$ *there exists an element* $t \in D$ *with* $\varphi(a) \cap [y) = [t)$.

In the following theorem we present a triple construction for decomposable MS-algebras.

Theorem 4.2. *Let* (M, D, φ) *be a decomposable* MS*-triple. Then*

$$L = \{(a, \varphi(a^0) \cup [x)) \mid a \in M, x \in D\}$$

is a decomposable MS*-algebra, if we define*

$$(a, \varphi(a^0) \cup [x)) \vee (b, \varphi(b^0) \cup [y)) = (a \vee b, (\varphi(a^0) \cup [x)) \wedge (\varphi(b^0) \cup [y))),$$

$$(a, \varphi(a^0) \cup [x)) \wedge (b, \varphi(b^0) \cup [y)) = (a \wedge b, (\varphi(a^0) \cup [x)) \vee (\varphi(b^0) \cup [y))),$$

$$(a, \varphi(a^0) \cup [x))^0 = (a^0, \varphi(a)),$$

$$1_L = (1, [1)),$$

$$0_L = (0, D).$$

Conversely, every decomposable MS*-algebra* L *can be associated with the decomposable* MS*-triple* $(L^{00}, D(L), \varphi(L))$, *where* $\varphi(L)(a) = [a^0) \cap D(L)$.

Proof. Let $(a, \varphi(a^0) \vee [x)), (b, \varphi(b^0) \vee [y)) \in L$. As φ is a $(0,1)$-homomorphism, we have

$$(a, \varphi(a^0) \vee [x)) \wedge (b, \varphi(b^0) \vee [y)) = (a \wedge b, \varphi((a \wedge b)^0) \vee [x \vee y)),$$

$$\begin{aligned}(a, \varphi(a^0) \vee [x)) \vee (b, \varphi(b^0) \vee [y)) &= (a \vee b, (\varphi(a^0) \vee [x)) \cap (\varphi(b^0) \vee [y)))\\ &= (a \vee b, \varphi((a \vee b)^0) \vee [t)), \qquad t \in D,\end{aligned}$$

and

$$\begin{aligned}&(\varphi(a^0) \vee [x)) \cap (\varphi(b^0) \vee [y)) =\\ &\qquad = (\varphi(a^0) \cap \varphi(b^0)) \vee (\varphi(a^0) \cap [y)) \vee (\varphi(b^0) \cap [x)) \vee ([x) \cap [y))\\ &\qquad = \varphi((a \vee b)^0) \vee [t)\,, \qquad t \in D,\end{aligned}$$

where $[t) = [q) \cup [p) \vee [x \vee y) = [q \wedge p \wedge (x \vee y))$, $\varphi(a^0) \cap [y) = [q)$ and $\varphi(b^0)\,[x) = [p)$, $p, q \in D$. This implies L is a sublattice of $M \times F_d(D)$.

Now we shall prove that L is an MS-algebra. Clearly,

$$\begin{aligned}(a, \varphi(a^0) \vee [x))^{00} &= (a^0, \varphi(a))^0\\ &= (a, \varphi(a^0))\\ &\geq (a, \varphi(a^0) \vee [x)),\end{aligned}$$

so the identity $(MS1)$ holds in L. Moreover, we have

$$\begin{aligned}\big[(a, \varphi(a^0) \vee [x)) \wedge (b, \varphi(b^0) \vee [y))\big]^0 &= (a \wedge b, \varphi((a \wedge b)^0) \vee [x \wedge y))^0\\ &= ((a \wedge b)^0, \varphi((a \wedge b)))\\ &= (a^0 \vee b^0, \varphi(a) \wedge \varphi(b))\\ &= (a^0, \varphi(a)) \vee (b^0, \varphi(b))\\ &= (a, \varphi(a^0) \vee [x))^0 \vee (b, \varphi(b^0) \vee [y))^0\end{aligned}$$

and $(1, [1))^0 = (0, D)$, thus the indentities $(MS2)$, $(MS3)$ are satisfied in L.

It remains to prove that L is decomposable. For every $(a, \varphi(a^0) \vee [x)) \in L_1$ we have

$$\begin{aligned}(a, \varphi(a^0) \vee [x)) &= (a, \varphi(a^0)) \wedge (1, [x))\\ &= (a, \varphi(a^0)\,[x))^{00} \wedge (1, [x)),\end{aligned}$$

where $(1, [x)) \in D(L_1)$. We have proved that L is a decomposable MS-algebra.

Conversely, let L be a decomposable MS-algebra. Then L^{00} is a de Morgan algebra and $D(L)$ is a filter of L which is indeed a distributive lattice with 1.

Let us define the mapping $\varphi(L) : L^{00} \to F(D(L))$ by

$$\varphi(L)(a) = \big[a^0\big) \cap D(L).$$

Obviously, $\varphi(L)$ is a $(0,1)$-homomorphism. Hence $(L^{00}, D(L), \varphi(L))$ is a decomposable MS-triple.

Now let us consider the mapping $\alpha : L \to L^{00}\varphi(L)F_d(D(L))$ defined by $\alpha(x) = (x^{00}, [x) \cap D(L))$. Then for each $(x^{00}, \varphi(L)(x^0) \vee [d)) \in L^{00}\varphi(L)F_d(D(L))$ we have $(x^{00}, \varphi(L)(x^0) \vee [d)) = (x^{00}, [x) \cap D(L)) = \alpha(x)$ and α is surjective.

To prove that α is injective, let $\alpha(x) = \alpha(y)$ for some $x, y \in L$. Then $(x^{00}, [x) \cap D(L)) = (y^{00}, [y) \cap D(L))$ implies that $x^{00} = y^{00}$ and $[x) \cap D(L) = [y) \cap D(L)$. Now using the condition (ii) of Definition 2.2 we get $x = y$, as required.

Finally, we have $(\alpha(x))^0 = (x^{00}, [x) \cap D(L))^0 = (x^0, [x^0) \cap D(L)) = \alpha(x^0)$ and also

$$\begin{aligned}
\alpha(x \wedge y) &= ((x \wedge y)^{00}, [x \wedge y) \cap D(L)) \\
&= (x^{00} \wedge y^{00}, ([x) \vee [y)) \cap D(L) \\
&= (x^{00} \wedge y^{00}, ([x) \cap D(L)) \vee ([y) \cap D(L))) \\
&= (x^{00}, [x) \cap D(L)) \wedge (y^{00}, [y) \cap D(L)) \\
&= \alpha(x) \wedge \alpha(y)
\end{aligned}$$

and

$$\begin{aligned}
\alpha(x \vee y) &= ((x \vee y)^{00}, [x \vee y) \cap D(L)) \\
&= (x^{00} \vee y^{00}, ([x) \wedge [y)) \cap D(L) \\
&= (x^{00} \vee y^{00}, ([x) \cap D(L)) \wedge ([y) \cap D(L))) \\
&= (x^{00}, [x) \cap D(L)) \vee (y^{00}, [y) \cap D(L)) \\
&= \alpha(x) \vee \alpha(y).
\end{aligned}$$

Hence α is the desired isomorphism. □

We shall say that the decomposable MS-algebra constructed in Theorem 4.2 is *associated* with the decomposable MS-triple (M, D, φ) and the construction of L described in Theorem 4.2 will be called a *decomposable MS-construction.*

Remark 4.3. *A class of decomposable MS-algebras and a similar triple construction of its members were also given by Badawy in* [1]. *However, his definition of decomposable MS-algebras is missing the condition (ii) (cf. Definition 2.2) without which we think the construction cannot be carried out.*

On the other hand, his triple construction of principal MS-algebras in [2] *which mimics the triple construction in* [9] *seems to be correct. However, we think that our imitation of the construction of* [9] *is better presented and we avoid the mistakes of the later parts of his work.*

Lemma 4.4. *Let L be a decomposable MS-algebra associated with the decomposable triple (M, D, φ). Then*

(i) $L^{00} = \{(a, \varphi(a^0)) \mid a \in M\}$;

(ii) $D(L) = \{(1, [x)) \mid x \in D\}$;

(iii) $D \cong D(L), M \cong L^{00}$.

Proof. (i) As $(a, \varphi(a^0) \vee [x))^{00} = (a^0, \varphi(a))^0 = (a, \varphi(a^0))$ for every $a \in M$, we have $L^{00} = \{(a, \varphi(a^0)) \mid a \in M\}$.

(ii) For every $x \in D$ $(1, [x))^0 = (1, \varphi(1^0) \vee [x))^0 = (0, \varphi(1)) = (0, D)$ holds. Hence $D(L) = \{(1, [x)) \mid x \in D\}$.

(iii) It is easy to check that $\psi : a \mapsto (a, \varphi(a^0))$ and $\chi : d \mapsto (1, [d))$ are desired isomorphisms of M and L^{00}, and of D and $D(L)$, respectively. ☐

Definition 4.5. *An isomorphism of decomposable MS-triples (M, D, φ) and (M_1, D_1, φ_1) is a pair (α, β) where α is an isomorphism of M and M_1, β is an isomorphism of D and D_1 and the diagram*

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & F(D) \arrow[d, "F(\beta)"] \\
M_1 \arrow[r, "\varphi_1"] & F(D_1)
\end{tikzcd}$$

commutes. ($F(\beta)$ is the isomorphism of $F(D)$ and $F(D_1)$ induced by β.)

Theorem 4.6. *Two decomposable MS-algebras are isomorphic if and only if their associated MS-triples are isomorphic.*

Proof. Let $(L^{00}, D(L), \varphi(L)), (M^{00}, D(M), \varphi(M))$ be decomposable MS-algebras such that $(L^{00}, D(L), \varphi(L)) \cong (M^{00}, D(M), \varphi(M))$.

Let $\tau : L \to M$ be an isomorphism. Let us consider the isomorphisms $\alpha : L^{00} \to M^{00}$ and $F(\beta) : F(D(L)) \to F(D(M))$ such that α is defined by $\alpha(x) = \tau(x)$ and β is defined by $\beta(A) = \{\tau(A) \mid a \in A\}$. Then we have

$$\begin{aligned}
\varphi(M)(\alpha(x)) &= \varphi(M)(\tau(x)) \\
&= \big[(\tau(x))^0\big) \cap D(M)
\end{aligned}$$

and

$$\begin{aligned}\beta(\varphi(L)(x)) &= \beta(\left[x^0\right) \cap D(L)) \\ &= \{\tau(y) \mid y \in \left[x^0\right) \cap D(L)\} \\ &= \left[(\tau(x))^0\right) \cap D(M),\end{aligned}$$

for every $x \in L^{00}$. So (α, β) is an isomorphism of decomposable triples $(L^{00}, D(L), \varphi(L))$ and $(M^{00}, D(M), \varphi(M))$.

Conversely, let us consider the mapping $g : L \to M$ defined by

$$g(a, \varphi(L)(a^0) \vee [x)) = (\alpha(a), F(\beta)([a) \cap D(L)) \vee [\beta(x))).$$

Now let $(a, \varphi(L)(a^0) \vee [x)) = (b, \varphi(L)(b^0) \vee [y))$. Then we have $a = b$ and $\varphi(L)(a^0) \vee [x) = \varphi(L)(b^0) \vee [y)$ and we immediately get $\alpha(a) = \alpha(b)$ and $([a) \cap D(L)) \vee [x) = ([b) \cap D(L)) \vee [y)$. Using $F(\beta)$ we obtain

$$(\alpha(a), F(\beta)([a) \cap D(L)) \vee [\beta(x))) = (\alpha(b), F(\beta)([b) \cap D(L)) \vee [\beta(y))).$$

Thus g is well-defined. One can also verify that g is a lattice isomorphism. From

$$\begin{aligned}g((a, \varphi(L)(a^0) \vee [x))^0) &= g(a^0, \varphi(L)(a)) \\ &= (\alpha(a^0), \varphi(M)(\alpha(a))) \\ &= (\alpha(a), \varphi(M)(\alpha(a))^0 \vee [\beta(x)))^0 \\ &= (g(a, \varphi(L)(a) \vee [x)))^0\end{aligned}$$

it follows that g is an MS-isomorphism and the proof is complete. □

5. CONCLUSION

In the presented work we give a simple triple construction of principal MS-algebras and we show that there is a one-to-one correspondence between principal MS-algebras and principal MS-triples. It is parallel to construction of principal p-algebras from principal triples in [9].

Moreover, we give a triple construction of a class of decomposable MS-algebras that includes the class of principal MS-algebras, which is a modification of the quadruple constructions by T. S. Blyth and J. C. Varlet ([5], [6]) and T. Katriňák and K. Mikula ([12]). Instead of Kleene algebras and filters $L^{\vee}$ used in the quadruples in [5], [6], [12], in our triples we use de Morgan algebras and filters $D(L)$.

The triple construction of principal MS-algebras is illustrated by two examples.

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