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LATTICES WITH RELATIVE STONE CONGRUENCE LATTICES II

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Abstract. In this companion paper to [4] we present an alternative characterization of lattices with relative Stone congruence lattices. For the first time, the (RS)-modularity is given by a condition that, unlike its versions known so far, does not include the quantification via congruences of a lattice. It is also closer to the already known characterization in the semi-discrete case [5], [7]. We similarly present an alternative characterization of lattices whose congruence lattices satisfy the identities (E_n) of T. Hecht and T. Katriňák [8] which describe the subvarieties of relative Stone Heyting algebras. This second result generalizes an old characterization of G. Grätzer and E.T. Schmidt of lattices with Boolean congruence lattices [3].

1. Introduction

G. Grätzer and E. T. Schmidt in [3] characterized lattices whose congruence lattices are Boolean thereby answering G. Birkhoff's problem [1] (cf. [1, Problem 39]). Their result (Theorem 4.3) has been presented in terms of weak projectivity of quotients of the lattice. In this paper we generalize Grätzer-Schmidt's result within the subvarieties of relative Stone Heyting algebras which are characterized by the identity

$$(E_n) \quad (x_0 * x_1) \vee (x_1 * x_2) \vee \ldots \vee (x_{n-1} * x_n) = 1$$

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introduced by T. Hecht and T. Katriňák [8]. Our characterization is given in Theorem 4.9. It is alternative to the one given in [4] and the version of the (E_n)-modularity we use here does not include the unpleasant quantification via congruences of a lattice as the version in [4].

We also give a new characterization, alternative to those in [4] and [5], of lattices with relative Stone congruence lattices (Theorem 3.9). As in [4], we use here the identity (RS) $(x*y)\vee(y*x)=1$ characterizing the relative Stone lattices within the variety of Heyting algebras but our version of the (RS)-modularity is given by a nicer symmetric condition that, for the first time, does not include the quantification via congruences of a lattice. Comparing to [4] and [5], it is also the most natural of the generalizations of already known characterization in the semi-discrete case [5], [7].

This is very much a companion paper to [4], and as such, a continuation of the papers [9], [5], [7] and [6] by T. Katriňák and the second author. The results are presented in terms of weak projectivity of quotients of a lattice.

2. Preliminaries

We denote by $\mathrm{Con}\,L$ the lattice of all congruence relations on a lattice L. The smallest and the largest congruence relation, respectively, are denoted by Δ and ∇. The lattice $\mathrm{Con}\,L$ satisfies the infinite distributivity law

$$\theta \wedge \bigvee(\alpha_i : i \in I) = \bigvee(\theta \wedge \alpha_i : i \in I)$$

for any $\theta, \alpha_i \in \mathrm{Con}\,L$ (cf. [2]). Consequently, for any $\alpha, \beta \in \mathrm{Con}\,L$ there exists a largest congruence $\delta \in \mathrm{Con}\,L$ such that $\alpha \wedge \delta \leq \beta$. It is clear that

$$\delta = \bigvee(\sigma : \alpha \wedge \sigma \leq \beta).$$

The congruence δ is known as the *relative pseudocomplement of α with respect to β* and is denoted by $\alpha * \beta$. It follows that $\langle \mathrm{Con}\,L, \vee, \wedge, *, \Delta, \nabla \rangle$ is a complete relatively pseudocomplemented lattice, thus a complete Heyting algebra.

The *bounded relative Stone lattices*, that is those bounded distributive lattices in which all intervals are Stone lattices, are characterized as Heyting algebras satisfying the identity

$$(RS) \quad (x*y) \vee (y*x) = 1$$

(cf. [10, 2.9. and 2.10]). T. Hecht and T. Katriňák [8] were the first to discover that the lattice of all subvarieties of the variety of bounded relative Stone lattices is isomorphic to the chain of type $\omega + 1$ and that a Heyting algebra L belongs to its n-th ($n \geq 2$) subvariety if and only if it satisfies the identity

$$(E_n) \quad (x_0 * x_1) \vee (x_1 * x_2) \vee \ldots \vee (x_{n-1} * x_n) = 1.$$

We note that the subvariety satisfying (E_2) is exactly the variety of all Boolean algebras.

Traditionally, we use the notation $a/b \to c/d$ for the weak projectivity of quotients of the lattice L (for the detailed definition see [2, Chapter 3]). By a *non-trivial* quotient a/b we mean that $b < a$ in L and by a *proper* subquotient $a'/b' \subset a/b$ we mean that $b \leq b' \leq a' \leq a$ but $a'/b' \neq a/b$. By a *prime* quotient a/b we mean that a covers b in L. We recall that if $a/b \to c/d$ and $(a, b) \in \theta$ for a congruence $\theta \in \operatorname{Con} L$ then also $(c, d) \in \theta$. The importance of the *weak projectivity of quotients of a lattice* in the description of lattice congruences comes from the following results.

Lemma 2.1. ([2, Theorem III.1.2] or [6, Lemma 1]) *For any principal congruence $\theta_{a,b} \in \operatorname{Con} L$,*

$$(c, d) \in \theta_{a,b},$$

$(d \leq c, b \leq a)$ if and only if there is a finite chain $d = y_0 \leq \ldots \leq y_m = c$ such that $a/b \to y_{i+1}/y_i$ for all $i \in \{0, \ldots, m-1\}$.

As in [4], we use the following abbreviation: for a lattice L and quotients $a/b, c/d$ of L, we shall say that *there are projections from the quotient a/b into a chain in c/d* or that *a/b has projections into a chain in c/d* if there is a finite chain $d = y_0 \leq \ldots \leq y_m = c$ such that $a/b \to y_{i+1}/y_i$ for all $i \in \{0, \ldots, m-1\}$. Then Lemma 2.1 can be rephrased such that $(c, d) \in \theta_{a,b}$ if and only if a/b has projections into a chain in c/d.

Lemma 2.2. ([11, 1.4], [6, Lemma 2]) *Let L be a lattice and $\theta, \varphi \in \operatorname{Con} L$. Then the relative pseudocomplement of θ with respect to φ is*

$$\theta * \varphi = \bigvee (\theta_{u,v}, (u, v) \in S),$$

where S is the set of all pairs of elements (u, v) $(u, v \in L)$ such that $u/v \to z/t$ and $(z, t) \in \theta$ implies $(z, t) \in \varphi$ for all $z, t \in L$.

Lemma 2.3. ([4, Corollary 2.3]) *Let L be a lattice, $\theta, \varphi \in \operatorname{Con} L$ and let $a, b \in L$, $b < a$.*

(i) *$(a, b) \in \theta * \varphi$ if and only if for every projection $a/b \to c/d$ $(c, d \in L)$, $(c, d) \in \theta$ yields $(c, d) \in \varphi$;*

(ii) $(a,b) \notin \theta{*}\varphi$ *if and only if there is a projection* $a/b \to c/d$ $(c,d \in L)$ *such that* $(c,d) \in \theta$ *and* $(c,d) \notin \varphi$.

3. Lattices whose congruence lattices are relative Stone

To compare our new characterizations of lattices with relative Stone congruence lattices with those already known from [5] and [4], we start with recalling the main concepts and the results of [5] and [4].

Definition 3.1. ([5], Definition 1) *Let* L *be a lattice,* $\pi \in \operatorname{Con} L$ *and* $a/b, u/v$ *be quotients of* L. *Then* L *is said to be* π-**almost weakly modular** *whenever* $a/b \to u/v$ *and* $(u,v) \notin \pi$ *imply the existence of a subquotient* $a_1/b_1 \subseteq a/b$ *with* $(a_1,b_1) \notin \pi$ *such that for every quotient* r/s *with* $a_1/b_1 \to r/s$ *and* $(r,s) \notin \pi$ *there exists a quotient* z/t *with* $r/s \to z/t$, $u/v \to z/t$ *and* $(z,t) \notin \pi$.

Definition 3.2. ([5], Definition 2) *Let* L *be a lattice and* $\theta, \pi \in \operatorname{Con} L$. *Then* θ *is said to be* π-**weakly separable** *if* $\pi \leq \theta$ *and for any* $a < b$ *in* L *there is a chain* $a = z_0 \leq z_1 \leq \ldots \leq z_m = b$ *such that for each* $i \in \{0, \ldots, m-1\}$ *either*

(i) $z_{i+1}/z_i \to u/v$ *and* $(u,v) \in \theta$ *imply* $(u,v) \in \pi$ *or*
(ii) *for every subquotient* $r/s \subseteq z_{i+1}/z_i$ *with* $(r,s) \notin \pi$, *there exists a quotient* u/v *with* $r/s \to u/v$ *and* $(u,v) \in \theta$, $(u,v) \notin \pi$.

Proposition 3.3. ([5], Theorem 2) Let L be a lattice. The lattice $\operatorname{Con} L$ is relatively Stone if and only if for every $\pi \in \operatorname{Con} L$ the following hold:

(1) L is π-almost weakly modular and
(2) every congruence of L is π-weakly separable.

The following concepts introduced in [4], alternative to those above, were already motivated by the symmetric identity (RS). The disadvantage of the version of the (RS)-weakly modularity used in [4] is the quantification of the given condition via arbitrary congruences $\theta, \pi \in \operatorname{Con} L$.

Definition 3.4. ([4, Definition 3.4]) Let L be a lattice. Let $a/b, u/v, z/t$ be quotients of L and let $\theta, \pi \in \operatorname{Con} L$. Then L is said to be **(RS)-weakly modular** whenever $a/b \to u/v$ and $a/b \to z/t$ with $(u,v) \in \theta$ and $(z,t) \in \pi$ imply that either one of (I) $(u,v) \in \pi$, (II) $(z,t) \in \theta$ holds or the following condition is satisfied:

(III) there are proper subquotients $a_1/b_1 \subset a/b$, $a_2/b_2 \subset a/b$ and quotients $u'/v', z'/t'$ such that $a_1/b_1 \to u'/v'$, $(u',v') \in \theta$, $(u',v') \notin \pi$ and $a_2/b_2 \to z'/t'$, $(z',t') \in \pi$, $(z',t') \notin \theta$.

Definition 3.5. ([4, Definition 3.5]) *Let L be a lattice and let θ, π be congruences of L. Then a pair θ, π is said to be* **(RS)-separable** *if for every $b < a$ in L there is a finite chain $b = x_0 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ one of the following conditions holds:*

(i) *for every proper subquotient $r_1/s_1 \subset x_{i+1}/x_i$ and every projection $r_1/s_1 \to u_1/v_1$, $(u_1, v_1) \in \theta$ implies $(u_1, v_1) \in \pi$;*
(ii) *for every proper subquotient $r_2/s_2 \subset x_{i+1}/x_i$ and every projection $r_2/s_2 \to u_2/v_2$, $(u_2, v_2) \in \pi$ implies $(u_2, v_2) \in \theta$.*

The characterization of lattices with relative Stone congruence lattices given in [4] was presented as follows:

Proposition 3.6. ([4, Theorem 3.6]) Let L be a lattice. Then $\operatorname{Con} L$ satisfies the identity

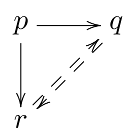
$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla$$

if and only if

(1) L is (RS)-weakly modular and
(2) every pair θ, π of congruences of L is (RS)-separable.

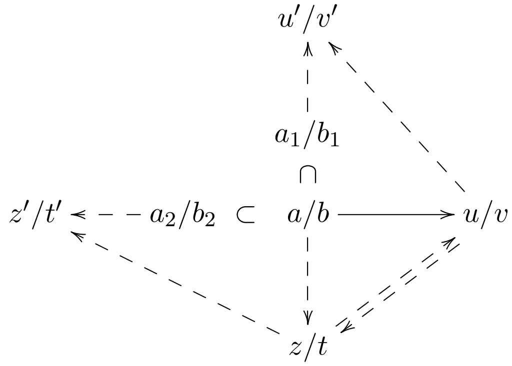
The characterizations of lattices with relative Stone congruence lattices given in Propositions 3.3 and 3.6 can both be essentially simplified if the lattice L is semi-discrete. Let us recall that a lattice L is called *semi-discrete* if between every two comparable elements of L there exists a finite maximal chain. (Every finite lattice is of course semi-discrete.) The following characterization in the semi-discrete case has already been given in [5].

Proposition 3.7. ([5], [4, Theorem 3.6]) *Let L be a semi-discrete lattice. Then $\operatorname{Con} L$ is a relative Stone lattice if and only if for any prime quotients p, q, r of L, the projections $p \to q$ and $p \to r$ imply that $q \to r$ or $r \to q$.*



We now introduce a new alternative version of the (RS)-modularity given by a nice symmetric condition that, for the first time, does not include the unpleasant quantification via arbitrary congruences of a lattice.

Comparing to [4] and [5], it is also the most natural of the generalizations of Proposition 3.7.



Definition 3.8. *A lattice L is* **(RS)-modular** *if for all non-trivial quotients $a/b, u/v, z/t$ of L, $a/b \to u/v$ and $a/b \to z/t$ yield that either one of the following conditions holds,*

(I) *there are projections from the quotient z/t into a chain in u/v,*

(II) *there are projections from the quotient u/v into a chain in z/t,*

or the following condition is satisfied:

(III) *there are proper subquotients $a_1/b_1 \subset a/b$, $a_2/b_2 \subset a/b$ and quotients $u'/v', z'/t'$ with $a_1/b_1 \to u'/v'$, $u/v \to u'/v'$, $(u', v') \notin \theta_{z,t}$ and $a_2/b_2 \to z'/t'$, $z/t \to z'/t'$, $(z', t') \notin \theta_{u,v}$.*

In our main result of this section we use the above new condition of (RS)-modularity together with the already used (in [4]) condition of (RS)-separability.

Theorem 3.9. *Let L be a lattice. Then* $\operatorname{Con} L$ *satisfies the identity*

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla$$

and so is a relative Stone lattice if and only if

(1) *L is (RS)-modular and*

(2) *every pair θ, π of congruences of L is (RS)-separable.*

Proof. For the necessity, let $\operatorname{Con} L$ satisfy the identity (RS). We note that all the quotients considered below are non-trivial. To show that L is (RS)-modular, let $a/b \to u/v$ and $a/b \to z/t$. Set

$$\theta := \theta_{u,v}, \ \pi := \theta_{z,t}.$$

Since $(a,b) \in (\theta * \pi) \vee (\pi * \theta)$, there exists a chain $b = x_0 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. We shall distinguish the following three cases.

(I) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$. So $(a,b) \in \theta * \pi$ and since $a/b \to u/v$ and $(u,v) \in \theta$, by Lemma 2.3(i) we get $(u,v) \in \pi$. As $\pi = \theta_{z,t}$, by Lemma 2.1 there are projections from the quotient z/t into a chain in u/v. So the condition (I) of Definition 3.8 holds.

(II) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \pi * \theta$. Then $(a,b) \in \pi * \theta$ and similarly as above one can obtain the condition (II) of Definition 3.8.

(III) Now assume that (I) and (II) do not hold so $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{j+1}, x_j) \notin \pi * \theta$ for some $i, j \in \{0, \ldots, m-1\}$ and proper quotients $x_{i+1}/x_i, x_{j+1}/x_j \subset a/b$. We set $a_1/b_1 := x_{i+1}/x_i$ and $a_2/b_2 := x_{j+1}/x_j$. Then $(a_1, b_1) \notin \theta * \pi$ and $(a_2, b_2) \notin \pi * \theta$ yield by Lemma 2.3(ii) that there exist projections $a_1/b_1 \to c_1/d_1$ with $(c_1, d_1) \in \theta, (c_1, d_1) \notin \pi$ and $a_2/b_2 \to c_2/d_2$ with $(c_2, d_2) \in \pi, (c_2, d_2) \notin \theta$. Since $(c_1, d_1) \in \theta_{u,v}$, there are projections from u/v into a chain in c_1/d_1. As $(c_1, d_1) \notin \pi$, there exists a subquotient $u'/v' \subseteq c_1/d_1$ such that $(u', v') \notin \pi$ and $u/v \to u'/v'$, $a_1/b_1 \to u'/v'$. Similarly, as $(c_2, d_2) \in \theta_{z,t}$, z/t has projections into a chain in c_2/d_2 and there is a subquotient $z'/t' \subseteq c_2/d_2$ such that $(z', t') \notin \theta$ and $z/t \to z'/t'$, $a_2/b_2 \to z'/t'$. We finally note that $(u', v') \notin \pi$ means $(u', v') \notin \theta_{z,t}$ and $(z', t') \notin \theta$ means $(z', t') \notin \theta_{u,v}$. We have proved (1).

The necessity of (2) was already shown in [4], but as the argument is short and we want this proof to be self-contained, we briefly repeat the argument. Let $\theta, \pi \in \operatorname{Con} L$ and $a, b \in L$, $b < a$. As $(a,b) \in (\theta * \pi) \vee (\pi * \theta)$, there exists a chain $b = x_0 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. If for $i \in \{0, \ldots m-1\}$ we have $(x_{i+1}, x_i) \in \theta * \pi$, then for every proper subquotient $r_1/s_1 \subset x_{i+1}/x_i$ also $(r_1, s_1) \in \theta * \pi$. Hence by Lemma 2.3(i), for every projection $r_1/s_1 \to u_1/v_1$, $(u_1, v_1) \in \theta$ implies $(u_1, v_1) \in \pi$. So the condition (i) of Definition 3.5 is satisfied. Analogously, if for $i \in \{0, \ldots m-1\}$ we have $(x_{i+1}, x_i) \in \pi * \theta$, then (ii) of Definition 3.5 is satisfied.

We will prove the sufficiency. Assume that (1) and (2) are satisfied and $\theta, \pi \in \operatorname{Con} L$, $a, b \in L$, $b < a$. By the (RS)-separability of θ, π, there is a chain $b = x_0 \leq \ldots \leq x_m = a$ such that (i) or (ii) of Definition 3.5 holds for each $i \in \{0, \ldots, m-1\}$. We shall distinguish the following two cases.

(a) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. Then $(a,b) \in (\theta * \pi) \vee (\pi * \theta)$. We show that the remaining case is impossible.

(b) Let (a) above does not hold. So there exists $i \in \{0, \ldots, m-1\}$ such that $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{i+1}, x_i) \notin \pi * \theta$. By Lemma 2.3(ii), there exist quotients $u/v, z/t$ and projections $x_{i+1}/x_i \to u/v$ with $(u, v) \in \theta$, $(u, v) \notin \pi$ and $x_{i+1}/x_i \to z/t$ with $(z, t) \in \pi$, $(z, t) \notin \theta$. Now we use that L is (RS)-modular. If (I) of Definition 3.8 holds with $a/b = x_{i+1}/x_i$, then z/t has projections into a chain in u/v. Since $(z, t) \in \pi$, this yields $(u, v) \in \pi$, a contradiction. In the same way, if (II) of Definition 3.8 holds, then $(z, t) \in \theta$, a contradiction.

We finally note that we do not need to consider (III) of Definition 3.8 in our case with $a/b = x_{i+1}/x_i$. The reason is that if there exists an element $x \in L$ such that $x_i < x < x_{i+1}$ (i.e. the quotient x_{i+1}/x_i is not prime) then for both the proper subquotients x/x_i and x_{i+1}/x of the quotient x_{i+1}/x_i, one of the conditions (i) respectively (ii) of Definition 3.5 applies, which gives us that $(x_{i+1}, x) \in \theta * \pi$ and $(x, x_i) \in \theta * \pi$ respectively $(x_{i+1}, x) \in \pi * \theta$ and $(x, x_i) \in \pi * \theta$; thus $(x_{i+1}, x_i) \in \theta * \pi$ respectively $(x_{i+1}, x_i) \in \pi * \theta$, which contradicts our assumption above that $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{i+1}, x_i) \notin \pi * \theta$. The remaining case is that the quotient x_{i+1}/x_i is prime in which case (III) of Definition 3.8 cannot occur for our quotient $a/b = x_{i+1}/x_i$ and its projections into the quotients u/v and z/t. The proof is complete. □

We finally note that in semi-discrete lattices the chains in our conditions can be considered maximal and so the quotients can be considered prime; hence there are no proper subquotients of considered quotients. This means that for semi-discrete lattices, the conditions (I) and (II) in Definition 3.8 can be immediately simplified and the condition (III) requiring the existence of certain proper subquotients of the considered quotient a/b cannot be satisfied and so can be omitted. Also, we note that for semi-discrete lattices, the condition of (RS)-separability is satisfied vacuously and can be omitted from the characterization. So in the semi-discrete case, Theorem 3.9 gives us immediately Proposition 3.7.

4. Lattices whose congruence lattices satisfy the identity (E_n)

We start with repeating the characterization of lattices L with Boolean congruence lattices due to G. Grätzer and E.T. Schmidt [3].

Definition 4.1. ([3], cf. also [2, Definition III.1.8]) *Let L be a lattice and let $a, b, c, d \in L$, $b < a, d < c$. Then L is called* **weakly modular**

if $a/b \to c/d$ implies the existence of a proper subquotient $a'/b' \subset a/b$ satisfying $c/d \to a'/b'$.

Definition 4.2. ([3], cf. also [2, p. 155]) *Let L be a lattice. A congruence θ of the lattice L is called* **separable** *if, for all $a, b \in L$, $b < a$ there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta$ or $(u, v) \in \theta$ for no proper subquotient $u/v \subset x_{i+1}/x_i$.*

Proposition 4.3. ([3], cf. [2, Theorem III.4.9]) *Let L be a lattice. Then* $\operatorname{Con} L$ *is Boolean if and only if L is weakly modular and all congruences of L are separable.*

Our aim in this section is to give a generalization of the Grätzer-Schmidt result within the subvarieties of all relative Stone Heyting algebras defined by the identities (E_n) $(n \geq 2)$ which is alternative to the one presented in [4]. We start by recalling our companion characterization in [4]. To sort out the terminology settings, we shall call the version of (E_n)-modularity presented in [4] **($\mathbf{E_n}$)-weakly modularity**.

Definition 4.4. ([4, Definition 4.4]) Let L be a lattice and $n \geq 2$. Let $a/b, u_1/v_1, \ldots, u_n/v_n$ be quotients of L and $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$. Then L is said to be **($\mathbf{E_n}$)-weakly modular** whenever

$$a/b \to u_j/v_j \quad \text{and} \quad (u_j, v_j) \in \theta_j, \; j = 1, \ldots, n$$

imply that either

- (I_j) there is $j \in \{1, \ldots, n\}$ with $(u_j, v_j) \in \theta_{j+1}$ or
- (II) for all $j \in \{1, \ldots, n\}$ there are proper subquotients $a_j/b_j \subset a/b$ and projections $a_j/b_j \to u'_j/v'_j$ with $(u'_j, v'_j) \in \theta_j$ and $(u'_j, v'_j) \notin \theta_{j+1}$.

Definition 4.5. ([4, Definition 4.5]) Let L be a lattice, $n \geq 2$ and let $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$. Then the (unordered)(n+1)-tuple $\theta_1, \ldots, \theta_{n+1}$ is said to be **($\mathbf{E_n}$)-separable** if for any $b < a$ there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ with the following property:

- (j) for every proper subquotient $a'/b' \subset x_{i+1}/x_i$ and every quotient u'_j/v'_j, $a'/b' \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$ imply $(u'_j, v'_j) \in \theta_{j+1}$.

Proposition 4.6. ([4, Theorem 4.6]) Let L be a lattice and $n \geq 2$. The congruence lattice $\operatorname{Con} L$ satisfies the identity (E_n) if and only if all of the following conditions hold:

1. $\operatorname{Con} L$ is relative Stone;

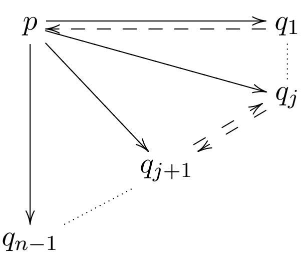
(2) L is (E_n)-weakly modular;
(3) every (n+1)-tuple of congruences on L is (E_n)-separable.

The presented characterization could again be essentially simplified in case the lattice L is semi-discrete. The result below was first proven in [5].

Proposition 4.7. ([5], [4, Corollary 5.1]) *Let L be a semi-discrete lattice. Then* $\operatorname{Con} L$ *is a relative Stone lattice satisfying the identity (E_n) $(n \geq 2)$ if and only if for any prime quotients $p, q_1, \ldots, q_{n-1}$ of L the projections $p \to q_k$ for $k \in \{1, \ldots, n-1\}$ imply that one of the following conditions holds:*

(i) *There exists $j \in \{1, \ldots, n-1\}$ such that there is a projection $q_j \to p$.*
(ii) *There exist $i, j \in \{1, \ldots, n-1\}$, $i \neq j$ such that there are projections $q_i \to q_j$ and $q_j \to q_i$.*

We note that by reordering the quotients above one can assume that $j = 1$ in the condition (i) and that $i = j + 1$ in the condition (ii) as the following figure indicates.



Our aim in this section is to introduce a new alternative version of the (E_n)-modularity given by a nicer condition that does not include the unpleasant quantification via arbitrary congruences of the considered lattice. Our result leads to a more natural generalization of Proposition 4.7 than our characterization given in Proposition 4.6.

Definition 4.8. *Let L be a lattice and $n \geq 2$. Let $a/b, u_1/v_1, \ldots, u_{n-1}/v_{n-1}$ be quotients of L. Then L is said to be* $(\mathbf{E_n})$**-modular** *whenever*

$$a/b \to u_i/v_i, \ i = 1, \ldots, n-1$$

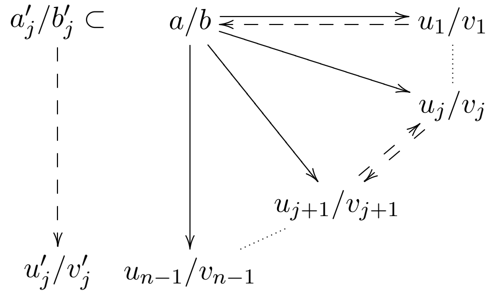
imply that either one of the following conditions holds,

(I) *u_1/v_1 has projections into a chain in a/b,*

(II) *there is* $j \in \{1, \dots, n-2\}$ *with* u_j/v_j *having projections into a chain in* u_{j+1}/v_{j+1} *and* u_{j+1}/v_{j+1} *having projections into a chain in* u_j/v_j,

or the following condition is satisfied:

(III) *there exist* $i, j \in \{1, \dots, n-1\}$ *with* $i \neq j$, *proper subquotients* $a'_i/b'_i \subset a/b$, $a'_j/b'_j \subset a/b$ *and quotients* $u'_i/v'_i, u'_j/v'_j$ *such that* $(u'_i, v'_i) \notin \theta_{u_j,v_j}$, $(u'_j, v'_j) \notin \theta_{u_i,v_i}$ *and* $a'_i/b'_i \to u'_i/v'_i$, $u_i/v_i \to u'_i/v'_i$ *and* $a'_j/b'_j \to u'_j/v'_j$, $u_j/v_j \to u'_j/v'_j$.



In our main result of this section we employ the above new condition of (E_n)-modularity together with the condition of (E_n)-separability used in [4].

Theorem 4.9. *Let* L *be a lattice and* $n \geq 2$. *The congruence lattice* $\operatorname{Con} L$ *satisfies the identity* (E_n) *if and only if all of the following conditions hold:*

(1) $\operatorname{Con} L$ *is relative Stone;*
(2) L *is* (E_n)*-modular;*
(3) *every (n+1)-tuple of congruences on* L *is* (E_n)*-separable.*

Proof. Let L be a lattice and $n \geq 2$. We assume that $\operatorname{Con} L$ satisfies the identity

$$(E_n) \quad (\theta_0 * \theta_1) \vee (\theta_1 * \theta_2) \vee \dots \vee (\theta_{n-1} * \theta_n) = \nabla.$$

We proved the necessity of (1) and (3) in [4], but as the arguments are short and we want this proof to be self-contained, we repeat the arguments briefly here. First we set $\theta = \theta_0 = \theta_2 = \dots$ and $\pi = \theta_1 = \theta_3 = \dots$ in the above congruence identity. Then we obtain that $\operatorname{Con} L$ satisfies the identity

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla.$$

Hence $\operatorname{Con} L$ is relative Stone and (1) holds.

To show the necessity of (3), let $\theta_0, \ldots, \theta_n$ be congruences of the lattice L. Since $(a,b) \in (\theta_0 * \theta_1) \vee (\theta_1 * \theta_2) \vee \ldots \vee (\theta_{n-1} * \theta_n)$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{0, \ldots, n-1\}$ with $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Let a'/b' be a proper subquotient of x_{i+1}/x_i and u'_j/v'_j be a quotient of L such that $a'/b' \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$. As $(a', b') \in \theta_j * \theta_{j+1}$, Lemma 2.3(ii) gives us $(u'_j, v'_j) \in \theta_{j+1}$ as required.

Most of the work is with proving the necessity of (2). Let $a/b \to u_i/v_i$ for $i \in \{1, \ldots, n-1\}$. We shall distinguish two cases.

(a) By the (RS)-modularity of L, there are $i, j \in \{1, \ldots, n-1\}$, $i \neq j$ such that for the projections $a/b \to u_i/v_i$, $a/b \to u_j/v_j$ the condition (III) of Definition 3.8 holds. It is easy to see that in this case the condition (III) of Definition 4.8 is immediately satisfied.

(b) The condition (a) above is not satisfied so that the (RS)-modularity of L yields that for every pair of projections $a/b \to u_i/v_i$, $a/b \to u_j/v_j$ where $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, one of the conditions (I), (II) of Definition 3.8 holds. Then there exists an ordering $i_1 \leq i_2 \leq \ldots \leq i_{n-1}$ of the set $\{1, \ldots, n-1\}$ such that there are projections from the quotient u_{i_j}/v_{i_j} into a chain in $u_{i_{j+1}}/v_{i_{j+1}}$ for $j = 1, \ldots, n-2$. Without loss of generality we can assume that $i_j = j$ for $j = 1, \ldots, n-1$. Now we set

$$\theta_0 := \theta_{a,b},\ \theta_1 := \theta_{u_1,v_1}, \ldots,\ \theta_{n-1} := \theta_{u_{n-1},v_{n-1}},\ \theta_n := \Delta.$$

Since $(a,b) \in (\theta_0 * \theta_1) \vee (\theta_1 * \theta_2) \vee \ldots \vee (\theta_{n-1} * \theta_n)$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there is $j \in \{0, \ldots, n-1\}$ with $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$.

We shall distinguish the following possible subcases of the case (b).

(b_1) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta_0 * \theta_1$. Then we have $(a,b) \in \theta_0 * \theta_1$. Since $(a,b) \in \theta_0$, by Lemma 2.3(i) we obtain $(a,b) \in \theta_1$. As $\theta_1 := \theta_{u_1,v_1}$, by Lemma 2.1 there are projections from u_1/v_1 into a chain in a/b.

(b_2) There exists $j \in \{1, \ldots, n-2\}$ such that for every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Then $(a,b) \in \theta_j * \theta_{j+1}$ whence $(u_j, v_j) \in \theta_j * \theta_{j+1}$ as $a/b \to u_j, v_j$. Since $(u_j, v_j) \in \theta_j$, it follows $(u_j, v_j) \in \theta_{j+1}$. Since we have $\theta_{j+1} := \theta_{u_{j+1},v_{j+1}}$, by Lemma 2.1 there are projections from u_{j+1}/v_{j+1} into a chain in u_j/v_j.

(b_3) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta_{n-1} * \theta_n$. Then we get $(a,b) \in \theta_{n-1} * \theta_n$, whence $(u_{n-1}, v_{n-1}) \in \theta_{n-1} * \theta_n$ as $a/b \to u_{n-1}/v_{n-1}$.

Now since $(u_{n-1}, v_{n-1}) \in \theta_{n-1}$, it follows $(u_{n-1}, v_{n-1}) \in \theta_n$, whence finally $u_{n-1} = v_{n-1}$, a contradiction.

$(\neg b_{1-3})$ Now we assume that none of the cases (b_1)-(b_3) is satisfied. So we assume that for every $j \in \{0, \dots, n-1\}$ there is $k_j \in \{0, \dots, m-1\}$ such that $(x_{k_j+1}, x_{k_j}) \notin \theta_j * \theta_{j+1}$. Let $a_j/b_j := x_{k_j+1}/x_{k_j} \subset a/b$ be the proper subquotient with $(a_j, b_j) \notin \theta_j * \theta_{j+1}$. By Lemma 2.3(ii) there exists a quotient c_j/d_j such that $a_j/b_j \to c_j/d_j$ and $(c_j, d_j) \in \theta_j$, $(c_j, d_j) \notin \theta_{j+1}$. As $\theta_j := \theta_{u_j,v_j}$ for all $j \in \{1, \dots, n-1\}$, by Lemma 2.1 there are projections from the quotient u_j/v_j into a chain in c_j/d_j. Let $j \in \{1, \dots, n-1\}$. Since $(c_j, d_j) \notin \theta_{j+1}$, there is a subquotient $u'_j/v'_j \subseteq c_j/d_j$ such that $(u'_j, v'_j) \notin \theta_{j+1}$ and $u_j/v_j \to u'_j/v'_j$. We note that $(u'_j, v'_j) \notin \theta_{j+1}$ means $(u'_j, v'_j) \notin \theta_{u_{j+1},v_{j+1}}$.

So in case (b) one of the following conditions is always satisfied:

- (I) There are projections from the quotient u_1/v_1 into a chain in a/b.
- (II) There exists $j \in \{1, \dots, n-2\}$ such that there are projections from the quotient u_j/v_j into a chain in u_{j+1}/v_{j+1} and there are projections from u_{j+1}/v_{j+1} into a chain in u_j/v_j.
- (III) There exist $i, j \in \{1, \dots, n-1\}$ with $i \neq j$, proper subquotients $a'_i/b'_i \subset a/b$, $a'_j/b'_j \subset a/b$ and quotients $u'_i/v'_i, u'_j/v'_j$ such that $(u'_i, v'_i) \notin \theta_{u_j,v_j}$, $(u'_j, v'_j) \notin \theta_{u_i,v_i}$ and $a'_i/b'_i \to u'_i/v'_i$, $u_i/v_i \to u'_i/v'_i$ and $a'_j/b'_j \to u'_j/v'_j$, $u_j/v_j \to u'_j/v'_j$.

We have proved (2).

Conversely, let the conditions (1)–(3) hold. Let $\theta_1, \dots, \theta_{n+1} \in \operatorname{Con} L$ and let $b < a$ in L. By the (E_n)-separability of $\theta_1, \dots, \theta_{n+1}$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for all $i \in \{0, \dots, m-1\}$ there exists $j \in \{1, \dots, n\}$ such that the condition (j) from Definition 4.5 is satisfied. We distinguish two cases.

(a) For every $i \in \{0, \dots, m-1\}$ there exists some $j \in \{1, \dots, n\}$ such that $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Then $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1})$ as required. We show that the remaining case is impossible.

(b) If (a) above does not hold, then there exists $i \in \{0, \dots, m-1\}$ such that for all $j \in \{1, \dots, n\}$, $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$. By Lemma 2.3(ii), for all $j \in \{1, \dots, n\}$ there exist quotients u_j/v_j and projections $x_{i+1}/x_i \to u_j/v_j$ with $(u_j, v_j) \in \theta_j$, $(u_j, v_j) \notin \theta_{j+1}$. Now we use that L is (E_n)-modular for the quotient $a/b = x_{i+1}/x_i$ and its projections $a/b \to u_2/v_2, \dots, a/b \to u_n/v_n$. But we shall also use that $a/b \to u_1/v_1$ and $(u_1, v_1) \notin \theta_2$.

If the condition (I) of Definition 4.8 holds, then there are projections from u_2/v_2 into a chain in a/b. As $(u_2, v_2) \in \theta_2$, we get $(a, b) \in \theta_2$. Since $a/b \to u_1/v_1$, we obtain $(u_1, v_1) \in \theta_2$, a contradiction.

If the condition (II) holds, then there is $j \in \{2, \dots, n-1\}$ such that there are projections from the quotient u_j/v_j into a chain in u_{j+1}/v_{j+1} and there are projections from u_{j+1}/v_{j+1} into a chain in u_j/v_j. As we have $(u_{j+1}, v_{j+1}) \in \theta_{j+1}$, we obtain $(u_j, v_j) \in \theta_{j+1}$, a contradiction.

We again note that we do not need to consider (III) of Definition 4.8 in our case. We recall that $a/b = x_{i+1}/x_i$. If there exists an element $x \in L$ such that $x_i < x < x_{i+1}$ (i.e. the quotient x_{i+1}/x_i is not prime) then for both the proper subquotients x/x_i and x_{i+1}/x of the quotient x_{i+1}/x_i, the condition (j) of Definition 4.5 applies for some $j \in \{1, \dots, n\}$. This gives us that $(x_{i+1}, x) \in \theta_j * \theta_{j+1}$ and $(x, x_i) \in \theta_j * \theta_{j+1}$ for some $j \in \{1, \dots, n\}$, which contradicts our assumption of case (b) that $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$ for all $j \in \{1, \dots, n\}$. The remaining case is that the quotient x_{i+1}/x_i is prime in which case (III) of Definition 4.8 cannot occur. The proof is complete. □

We again finally note that in semi-discrete lattices, the conditions (I) and (II) in Definition 4.8 can be immediately simplified and the condition (III) can be omitted. The condition of (E_n)-separability is satisfied vacuously and can be omitted from the characterization. So in the semi-discrete case, Theorem 4.9 gives us immediately Proposition 4.7. We emphasize this as an another important advantage of our new characterization in Theorem 4.9, comparing to the one in Proposition 4.6, because in [4] we needed a rather long proof to derive Proposition 4.7 from Proposition 4.6.

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