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LATTICES WITH RELATIVE STONE CONGRUENCE LATTICES

DANIELA GUFFOVÁ AND MIROSLAV HAVIAR

Abstract. We give a new characterization of lattices with relative Stone congruence lattices and we generalize, within the subvarieties of relative Stone Heyting algebras, a characterization of G. Grätzer and E.T. Schmidt of lattices with Boolean congruence lattices.

1. Introduction

The basic fact about the congruence lattices of lattices is that they are distributive and (relatively) pseudocomplemented, hence they can be investigated as Heyting algebras. It is natural to characterize lattices whose congruence lattices satisfy identities formulated in terms of (relative) pseudocomplement.

In [3], G. Grätzer and E. T. Schmidt characterized those lattices whose congruence lattices are Boolean thereby answering G. Birkhoff's problem (cf. [1, Problem 39]). Their result (Proposition 4.3) has been presented in terms of weak projectivity of quotients of the lattice. Our aim in this paper is to generalize Grätzer-Schmidt's result within the subvarieties of relative Stone Heyting algebras which are characterized by the identity

$$(E_n) \quad (x_1 * x_2) \vee (x_2 * x_3) \vee \ldots \vee (x_n * x_{n+1}) = 1.$$

This is done in Theorem 4.6. Since the identity (E_2) characterizes the variety of Boolean algebras, as a consequence of our result we obtain a new characterization of lattices with Boolean congruence lattices (Corollary 4.7).

This is a continuation of the previous papers [8], [4], [6] and [5] by T. Katriňák and the second author. We in addition give a new characterization, alternative to that in [4], of lattices with relative Stone congruence lattices (Theorem 3.6). Here we use the identity (RS) $(x * y) \vee (y * x) = 1$ characterizing the relative Stone lattices within the variety of Heyting algebras. Our results can be much simplified for semi-discrete lattices and this is done in Section 5.

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2. Preliminaries

Let $\operatorname{Con} L$ denote the lattice of all congruence relations on a lattice L with Δ and ∇, the smallest and the largest congruence relation, respectively. It is well known (cf. [2]) that $\operatorname{Con} L$ satisfies the infinite distributivity law

$$\theta \wedge \bigvee(\alpha_i : i \in I) = \bigvee(\theta \wedge \alpha_i : i \in I)$$

for any $\theta, \alpha_i \in \operatorname{Con} L$. It follows that for any $\alpha, \beta \in \operatorname{Con} L$ there exists a largest congruence $\delta \in \operatorname{Con} L$ such that $\alpha \wedge \delta \leq \beta$. Obviously, $\delta = \bigvee(\sigma : \alpha \wedge \sigma \leq \beta)$. The congruence δ is called the *relative pseudocomplement of* α *with respect to* β and is denoted by $\alpha * \beta$. Therefore $\langle \operatorname{Con} L, \vee, \wedge, *, \Delta, \nabla \rangle$ is a complete relatively pseudocomplemented lattice, that is, a complete Heyting algebra.

We recall that an algebra $\langle H, \vee, \wedge, *, 0, 1 \rangle$ of type (2, 2, 2, 0, 0) is a *Heyting algebra* if it satisfies, for all $x, y, z \in H$:
(H1) $\langle H, \vee, \wedge \rangle$ is a distributive lattice,
(H2) $x \wedge 0 = 0$,
(H3) $x * x = 1$,
(H4) $(x * y) \wedge y = y$, $x \wedge (x * y) = x \wedge y$,
(H5) $x * (y \wedge z) = (x * y) \wedge (x * z)$, $(x \vee y) * z = (x * z) \wedge (y * z)$.

Within the variety of bounded distributive lattices, the *bounded relative Stone lattices*, that is those in which all intervals are Stone lattices, are characterized exactly as Heyting algebras satisfying the identity (RS) $(x*y) \vee (y*x) = 1$ (see [9, 2.9. and 2.10]). In [7] it was shown that the lattice of all subvarieties of the variety of bounded relative Stone lattices is isomorphic to the chain of type $\omega + 1$ and that a Heyting algebra L belongs to the n-th ($n \geq 2$) subvariety if and only if it satisfies the identity

$$(E_n) \quad (x_1 * x_2) \vee (x_2 * x_3) \vee \ldots \vee (x_n * x_{n+1}) = 1.$$

The subvariety satisfying (E_2) is exactly the variety of all Boolean algebras.

We shall use the notation $a/b \to c/d$ for the weak projectivity of quotients of the lattice L (for the detailed definition see [2, Chapter 3]). By a *non-trivial* quotient a/b we mean that $b < a$ in L and by a *proper* subquotient $a'/b' \subset a/b$ we mean that $b \leq b' \leq a' \leq a$ but $a'/b' \neq a/b$. We note a basic fact that if $a/b \to c/d$ and $(a, b) \in \theta$ for a congruence $\theta \in \operatorname{Con} L$ then also $(c, d) \in \theta$. The importance of the *weak projectivity of quotients of a lattice* in the description of lattice congruences is given by the following three results.

Lemma 2.1 ([2, Theorem III.1.2] or [5, Lemma 1]). *For any principal congruence* $\theta_{a,b} \in \operatorname{Con} L$,

$$(c, d) \in \theta_{a,b}$$

$(d \leq c, b \leq a)$ *if and only if there is a finite chain* $d = y_0 \leq \ldots \leq y_m = c$ *such that* $a/b \to y_{i+1}/y_i$ *for all* $i \in \{0, \ldots, m-1\}$.

Lemma 2.2 ([10, 1.4] or [5, Lemma 2]). *Let L be a lattice and $\theta, \varphi \in \operatorname{Con} L$. Then the relative pseudocomplement of θ with respect to φ is*

$$\theta * \varphi = \bigvee(\theta_{u,v} : (u, v) \in S),$$

where S is the set of all pairs of elements (u, v) $(u, v \in L)$ such that $u/v \to z/t$ and $(z, t) \in \theta$ imply $(z, t) \in \varphi$ for all $z, t \in L$.

Corollary 2.3. *Let L be a lattice, $\theta, \varphi \in \operatorname{Con} L$ and let $a, b \in L$, $b < a$.*

(i) *$(a, b) \in \theta * \varphi$ if and only if for every projection $a/b \to c/d$ $(c, d \in L)$, $(c, d) \in \theta$ yields $(c, d) \in \varphi$;*
(ii) *$(a, b) \notin \theta * \varphi$ if and only if there is a projection $a/b \to c/d$ $(c, d \in L)$ such that $(c, d) \in \theta$ and $(c, d) \notin \varphi$.*

Proof. It is sufficient to show (i). If $(a, b) \in \theta * \varphi$ and $a/b \to c/d$, then $(c, d) \in \theta * \varphi$. Now $(c, d) \in \theta$ and the definition of $\theta * \varphi$ (see the beginning of Section 2) imply immediately that $(c, d) \in \varphi$. Conversely, suppose that for every projection $a/b \to c/d$ $(c, d \in L)$, $(c, d) \in \theta$ yields $(c, d) \in \varphi$. Then $(a, b) \in S$ for the set S from Lemma 2.2, hence $(a, b) \in \theta * \varphi$. □

3. Lattices with relative Stone congruence lattices

In this section we give a new characterization of lattices with relative Stone congruence lattices which is alternative to that in [4] (see Proposition 3.3 below). To obtain our characterization in [4], the bounded relative Stone lattices were considered as pseudocomplemented lattices in which every interval is Stone, that is, satisfies the identity $x^* \vee x^{**} = 1$. In our characterization here we understand the bounded relative Stone lattices as Heyting algebras satisfying the identity (RS) $(x * y) \vee (y * x) = 1$.

We start with recalling the main definitions and the result of [4].

Definition 3.1 ([4], Definition 1). *Let L be a lattice, $\pi \in \operatorname{Con} L$ and $a/b, u/v$ quotients of L. Then L is said to be π-**almost weakly modular** whenever $a/b \to u/v$ and $(u, v) \notin \pi$ imply the existence of a subquotient $a_1/b_1 \subseteq a/b$ with $(a_1, b_1) \notin \pi$ such that for every quotient r/s with $a_1/b_1 \to r/s$ and $(r, s) \notin \pi$ there exists a quotient z/t with $r/s \to z/t$, $u/v \to z/t$ and $(z, t) \notin \pi$.*

Definition 3.2 ([4], Definition 2). *Let L be a lattice and $\theta, \pi \in \operatorname{Con} L$. Then θ is π-**weakly separable** if $\pi \leq \theta$ and for any $a < b$ in L there is a chain $a = z_0 \leq z_1 \leq \ldots \leq z_m = b$ such that for each $i \in \{0, \ldots, m-1\}$ either*

(i) *$z_{i+1}/z_i \to u/v$ and $(u, v) \in \theta$ imply $(u, v) \in \pi$ or*
(ii) *for every subquotient $r/s \subseteq z_{i+1}/z_i$ with $(r, s) \notin \pi$, there exists a quotient u/v with $r/s \to u/v$ and $(u, v) \in \theta$, $(u, v) \notin \pi$.*

Proposition 3.3 ([4], Theorem 2). *Let L be a lattice. The lattice $\operatorname{Con} L$ is relatively Stone if and only if for every $\pi \in \operatorname{Con} L$ the following hold:*

(1) *L is π-almost weakly modular and*
(2) *every congruence of L is π-weakly separable.*

Our following definitions, which are alternative to those above, are motivated by the symmetric identity (RS).

Definition 3.4. *Let L be a lattice. Let $a/b, u/v, z/t$ be quotients of L and let $\theta, \pi \in \operatorname{Con} L$. Then L is said to be* **(RS)-weakly modular** *whenever $a/b \to u/v$ and $a/b \to z/t$ with $(u,v) \in \theta$ and $(z,t) \in \pi$ imply that either one of* (I) *$(u,v) \in \pi$,* (II) *$(z,t) \in \theta$ holds or the following condition is satisfied:*

(III) *there are proper subquotients $a_1/b_1 \subset a/b$, $a_2/b_2 \subset a/b$ and quotients $u'/v', z'/t'$ such that $a_1/b_1 \to u'/v'$, $(u',v') \in \theta$, $(u',v') \notin \pi$ and $a_2/b_2 \to z'/t'$, $(z',t') \in \pi$, $(z',t') \notin \theta$.*

Definition 3.5. *Let L be a lattice and let θ, π be congruences of L. Then the pair θ, π is said to be* **(RS)-separable** *if for every $b < a$ in L there is a finite chain $b = x_0 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ one of the following conditions holds:*

(i) *for every proper subquotient $r_1/s_1 \subset x_{i+1}/x_i$ and every projection $r_1/s_1 \to u_1/v_1$ $(u_1, v_1 \in L)$, $(u_1,v_1) \in \theta$ implies $(u_1,v_1) \in \pi$;*
(ii) *for every proper subquotient $r_2/s_2 \subset x_{i+1}/x_i$ and every projection $r_2/s_2 \to u_2/v_2$ $(u_2, v_2 \in L)$, $(u_2,v_2) \in \pi$ implies $(u_2,v_2) \in \theta$.*

Now we can present our alternative characterization of lattices with relative Stone congruence lattices.

Theorem 3.6. *Let L be a lattice. Then $\operatorname{Con} L$ satisfies the identity*

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla$$

if and only if

(1) *L is (RS)-weakly modular and*
(2) *every pair θ, π of congruences of L is (*RS*)-separable.*

Proof. First we shall prove the necessity. Let $\operatorname{Con} L$ satisfy the identity (RS). We shall show that L is (RS)-weakly modular. All the quotients considered below are non-trivial. Let $\theta, \pi \in \operatorname{Con} L$ and let $a/b \to u/v$ and $a/b \to z/t$ with $(u,v) \in \theta$, $(z,t) \in \pi$.

Since $(a,b) \in (\theta * \pi) \vee (\pi * \theta)$, there exists a chain $b = x_0 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. We shall distinguish the following cases.

(i) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$. Then $(a,b) \in \theta * \pi$ and as $a/b \to u/v$ and $(u,v) \in \theta$, by Corollary 2.3(i) we obtain $(u,v) \in \pi$. So the condition (I) of Definition 3.4 holds.

(ii) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \pi * \theta$. Then $(a,b) \in \pi * \theta$ and analogously as above we get the condition (II) of Definition 3.4.

(iii) Now assume that (i) and (ii) above do not hold so $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{j+1}, x_j) \notin \pi * \theta$ for some $i, j \in \{0, \ldots, m-1\}$ and proper quotients $x_{i+1}/x_i, x_{j+1}/x_j \subset a/b$. Set $a_1/b_1 := x_{i+1}/x_i$ and $a_2/b_2 := x_{j+1}/x_j$. Then $(a_1, b_1) \notin \theta * \pi$ and $(a_2, b_2) \notin \pi * \theta$ yield by Corollary 2.3(ii) that there exist quotients $u'/v', z'/t'$ such that $a_1/b_1 \to u'/v'$, $(u', v') \in \theta$, $(u', v') \notin \pi$ and $a_2/b_2 \to z'/t'$, $(z', t') \in \pi$, $(z', t') \notin \theta$. We have proven (1).

To show (2), let $\theta, \pi \in \operatorname{Con} L$ and $a, b \in L$, $b < a$. As $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$, there exists a chain $b = x_0 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. If for $i \in \{0, \ldots, m-1\}$ we have $(x_{i+1}, x_i) \in \theta * \pi$, then for every proper subquotient $r_1/s_1 \subset x_{i+1}/x_i$ also $(r_1, s_1) \in \theta * \pi$, whence by Corollary 2.3(i) for every projection $r_1/s_1 \to u_1/v_1$, $(u_1, v_1) \in \theta$ implies $(u_1, v_1) \in \pi$. So the condition (i) of Definition 3.5 is satisfied. Analogously, if for $i \in \{0, \ldots, m-1\}$ we have $(x_{i+1}, x_i) \in \pi * \theta$, then for every proper subquotient $r_2/s_2 \subset x_{i+1}/x_i$ and every projection $r_2/s_2 \to u_2/v_2$, $(u_2, v_2) \in \pi$ implies $(u_2, v_2) \in \theta$, so (ii) of Definition 3.5 holds. This gives (2).

Now let (1) and (2) hold and $\theta, \pi \in \operatorname{Con} L$, $a, b \in L$, $b < a$. By the (*RS*)-separability of θ, π, there is a chain $b = x_0 \leq \ldots \leq x_m = a$ such that (i) or (ii) of Definition 3.5 holds for each $i \in \{0, \ldots, m-1\}$. We distinguish two cases.

(a) For every $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta * \pi$ or $(x_{i+1}, x_i) \in \pi * \theta$. Then $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$. We show that the remaining case is impossible.

(b) Now assume that (a) above does not hold. So there is $i \in \{0, \ldots, m-1\}$ such that $(x_{i+1}, x_i) \notin \theta * \pi$ and $(x_{i+1}, x_i) \notin \pi * \theta$. By Corollary 2.3(ii), there exist quotients $u/v, z/t$ and projections $x_{i+1}/x_i \to u/v$ with $(u, v) \in \theta$, $(u, v) \notin \pi$ and $x_{i+1}/x_i \to z/t$ with $(z, t) \in \pi$, $(z, t) \notin \theta$. Now we use that L is (*RS*)-weakly modular. If (I) of Definition 3.4 holds, then $(u, v) \in \pi$, a contradiction. Analogously, if (II) of Definition 3.4 holds, then $(z, t) \in \theta$, a contradiction.

Now let (III) of Definition 3.4 hold. So there are proper subquotients $a_1/b_1 \subset x_{i+1}/x_i$, $a_2/b_2 \subset x_{i+1}/x_i$ and projections $a_1/b_1 \to u'/v'$, $(u', v') \in \theta$, $(u', v') \notin \pi$ and $a_2/b_2 \to z'/t'$, $(z', t') \in \pi$, $(z', t') \notin \theta$. If (i) of Definition 3.5 holds, then $a_1/b_1 \to u'/v'$ and $(u', v') \in \theta$ gives $(u', v') \in \pi$, a contradiction. Analogously, if (ii) of Definition 3.5 holds, then $a_2/b_2 \to z'/t'$ and $(z', t') \in \pi$ gives $(z', t') \in \theta$, a contradiction. The proof is complete. □

4. Lattices whose relative Stone congruence lattices satisfy the identity (E_n)

G. Grätzer and E. T. Schmidt in [3] characterized lattices L whose congruence lattice is Boolean by introducing the concepts of a *weakly modular* lattice and of a *separable* congruence.

Definition 4.1 ([3], cf. also [2, Definition III.1.8]). *Let L be a lattice and let $a, b, c, d \in L$, $b < a, d < c$. Then L is called* **weakly modular** *if $a/b \to c/d$ implies the existence of a proper subquotient $a'/b' \subset a/b$ satisfying $c/d \to a'/b'$.*

Definition 4.2 ([3], cf. also [2, p. 155]). *Let L be a lattice. A congruence θ of the lattice L is called* **separable** *if, for all $a, b \in L$, $b < a$ there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for each $i \in \{0, \ldots, m-1\}$, $(x_{i+1}, x_i) \in \theta$ or $(u, v) \in \theta$ for no proper subquotient $u/v \subset x_{i+1}/x_i$.*

Here is the Grätzer-Schmidt characterization.

Proposition 4.3 ([3], cf. also [2, Theorem III.4.9]). *Let L be a lattice. Then $\mathrm{Con}\, L$ is Boolean if and only if L is weakly modular and all congruences of L are separable.*

Our aim in this section is to generalize their result within the subvarieties of all relative Stone algebras defined by the identities (E_n), $n \geq 2$. We recall that the variety of Boolean algebras is the subvariety defined by the identity (E_2). The following definitions generalize the Definitions 4.1 and 4.2.

Definition 4.4. *Let L be a lattice and $n \geq 2$. Let $a/b, u_1/v_1, \ldots, u_n/v_n$ be quotients of L and $\theta_1, \ldots, \theta_{n+1} \in \mathrm{Con}\, L$. Then L is said to be* **$(\boldsymbol{E}_{\mathbf{n}})$-modular** *whenever*

$$a/b \to u_j/v_j \quad \text{and} \quad (u_j, v_j) \in \theta_j,\ j = 1, \ldots, n$$

imply that either

- (I_j) *there is $j \in \{1, \ldots, n\}$ with $(u_j, v_j) \in \theta_{j+1}$ or*
- (II) *for all $j \in \{1, \ldots, n\}$ there are proper subquotients $a_j/b_j \subset a/b$ and projections $a_j/b_j \to u'_j/v'_j$ with $(u'_j, v'_j) \in \theta_j$ and $(u'_j, v'_j) \notin \theta_{j+1}$.*

Definition 4.5. *Let L be a lattice, $n \geq 2$ and $\theta_1, \ldots, \theta_{n+1} \in \mathrm{Con}\, L$. Then the (unordered)$(n+1)$-tuple $\theta_1, \ldots, \theta_{n+1}$ is said to be* **$(\boldsymbol{E}_{\mathbf{n}})$-separable** *if for any $b < a$ there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ with the following property:*

- (j) *for every proper subquotient $a'/b' \subset x_{i+1}/x_i$ and every quotient u'_j/v'_j, $a'/b' \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$ imply $(u'_j, v'_j) \in \theta_{j+1}$.*

The main result of this section is the following characterization of lattices L whose congruence lattice satisfies the identity (E_n).

Theorem 4.6. *Let L be a lattice and $n \geq 2$. The congruence lattice $\mathrm{Con}\, L$ satisfies the identity (E_n) if and only if all of the following conditions hold:*

(1) *$\mathrm{Con}\, L$ is relative Stone;* (2) *L is (E_n)-modular;*
(3) *every $(n+1)$-tuple of congruences $\theta_1, \ldots, \theta_{n+1}$ on L is (E_n)-separable.*

Proof. Let L be a lattice and $n \geq 2$. We assume that $\mathrm{Con}\, L$ satisfies the identity

$$(E_n) \quad (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1}) = \nabla.$$

Let us set $\theta = \theta_1 = \theta_3 = \ldots$ and $\pi = \theta_2 = \theta_4 = \ldots$ in the above congruence identity. Then we obtain that $\operatorname{Con} L$ satisfies the identity

$$(RS) \qquad (\theta * \pi) \vee (\pi * \theta) = \nabla.$$

Hence $\operatorname{Con} L$ is relative Stone and (1) holds.

Now we prove (2). All the quotients considered below are non-trivial. Let $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$ and let $a/b \to u_j/v_j$ with $(u_j, v_j) \in \theta_j$ for $j = 1, \ldots, n$.

Since $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1})$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ with $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$.

We shall distinguish the following two cases.

(i) There is $j \in \{1, \ldots, n\}$ such that for every $i \in \{0, \ldots, m-1\}$ we have $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Then $(a, b) \in \theta_j * \theta_{j+1}$ and since $a/b \to u_j/v_j$ and $(u_j, v_j) \in \theta_j$, by Corollary 2.3(i) we obtain $(u_j, v_j) \in \theta_{j+1}$. So the condition (I_j) of Definition 4.4 holds.

(ii) Now assume that (i) above does not hold, so for all $j \in \{1, \ldots, n\}$ there is $i_j \in \{0, \ldots, m-1\}$ with $(x_{i_j+1}, x_{i_j}) \notin \theta_j * \theta_{j+1}$. Let us denote by a_j/b_j the proper subquotient $x_{i_j+1}/x_{i_j} \subset a/b$, $j \in \{1, \ldots, n\}$. Then for each $j \in \{1, \ldots, n\}$, $(a_j, b_j) \notin \theta_j * \theta_{j+1}$ yields by Corollary 2.3(ii) that there exists a projection $a_j/b_j \to u'_j/v'_j$ with $(u'_j, v'_j) \in \theta_j$ and $(u'_j, v'_j) \notin \theta_{j+1}$. So the condition (II) of Definition 4.4 holds. We have proven (2).

Now we will prove (3). Let $\theta_1, \ldots, \theta_{n+1}$ be congruences of the lattice L and $b < a$ in L. Since $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1})$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for every $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ with $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$.

Let $i \in \{0, \ldots, m-1\}$ and $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$ for some $j \in \{1, \ldots, n\}$. Let a'/b' be a proper subquotient of x_{i+1}/x_i and u'_j/v'_j be a quotient of L such that $a'/b' \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$. As $(a', b') \in \theta_j * \theta_{j+1}$, Corollary 2.3(i) gives us $(u'_j, v'_j) \in \theta_{j+1}$ as required. We have shown the necessity of (3).

Conversely, let the conditions (1)–(3) hold. Let $\theta_1, \ldots, \theta_{n+1} \in \operatorname{Con} L$ and let $b < a$ in L. By the (E_n)-separability of $\theta_1, \ldots, \theta_{n+1}$, there exists a chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ such that for all $i \in \{0, \ldots, m-1\}$ there exists $j \in \{1, \ldots, n\}$ such that the condition (j) from Definition 4.5 is satisfied. We distinguish two cases.

(a) For every $i \in \{0, \ldots, m-1\}$ there exists some $j \in \{1, \ldots, n\}$ such that $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$. Then $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1})$ as required. We show that the remaining case is impossible.

(b) Assume that (a) above does not hold. So there exists $i \in \{0, \ldots, m-1\}$ such that for all $j \in \{1, \ldots, n\}$, $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$. By Corollary 2.3(ii), there exist quotients u_j/v_j and projections $x_{i+1}/x_i \to u_j/v_j$ with $(u_j, v_j) \in \theta_j$, $(u_j, v_j) \notin \theta_{j+1}$ for all $j \in \{1, \ldots, n\}$. Now we use that L is (E_n)-modular. If the condition (I_j) of Definition 4.4 holds, then there exists $j \in \{1, \dots, n\}$ with $(u_j, v_j) \in \theta_{j+1}$, a contradiction.

Now let (II) of Definition 4.4 hold. So for all $j \in \{1, \dots, n\}$ there are proper subquotients $a_j/b_j \subset x_{i+1}/x_i$ and projections $a_j/b_j \to u'_j/v'_j$ such that $(u'_j, v'_j) \in \theta_j$, $(u'_j, v'_j) \notin \theta_{j+1}$. Since the condition (j) from Definition 4.5 is satisfied for certain $j \in \{1, \dots, n\}$, $a_j/b_j \to u'_j/v'_j$ and $(u'_j, v'_j) \in \theta_j$ give $(u'_j, v'_j) \in \theta_{j+1}$, a contradiction. The proof is complete. □

As a consequence we derive a new characterization of lattices with Boolean congruence lattices equivalent to the one by G. Grätzer and E.T. Schmidt given in Proposition 4.3.

Corollary 4.7. *Let L be a lattice. Then $\operatorname{Con} L$ is Boolean if and only if the following conditions hold:*

(i) *L is (E_2)-modular and*
(ii) *every triple of congruences $\theta_1, \theta_2, \theta_3$ on L is (E_2)-separable.*

5. Semi-discrete case

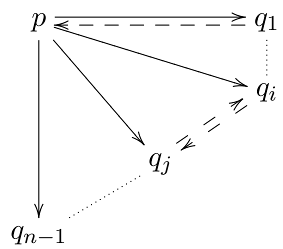
Our main results, Theorem 4.6 and Theorem 3.6, can both be essentially simplified if the lattice L is semi-discrete. Let us recall that a lattice L is called *semi-discrete* if between every two comparable elements of L there exists a finite maximal chain. Every finite lattice is clearly semi-discrete.

Hence in semi-discrete lattices the chains in our conditions can always be considered maximal and the quotients can be considered prime; thus there are no proper subquotients of considered quotients. This means that for semi-discrete lattices L, the conditions of (E_n)-separability and of (RS)-separability in Theorems 4.6 and 3.6, respectively are satisfied vacuously and can be omitted from the characterizations analogously as in the semi-discrete case in [4]. Similarly, the condition (II) in Definition 4.4 and the condition (III) in Definition 3.4 requiring the existence of certain proper subquotients of the considered quotients cannot be satisfied and so can be omitted.

Using these simplifications, we can derive from Theorem 4.6 the characterization of semi-discrete lattices with relative Stone congruence lattices satisfying the identity (E_n) ($n \geq 2$) published in [4, p. 89]:

Corollary 5.1. *Let L be a semi-discrete lattice. Then $\operatorname{Con} L$ is a relative Stone lattice satisfying the identity (E_n) ($n \geq 2$) if and only if for any prime quotients $p, q_1, \dots, q_{n-1}$ of L the projections $p \to q_k$ for $k \in \{1, \dots, n-1\}$ imply that one of the following conditions holds:*

(i) *There exists $j \in \{1, \dots, n-1\}$ such that there is a projection $q_j \to p$.*
(ii) *There exist $i, j \in \{1, \dots, n-1\}$, $i \neq j$ such that there are projections $q_i \to q_j$ and $q_j \to q_i$.*



Proof. Let L be a semi-discrete lattice and $n \geq 2$. We first prove the necessity. Let $\mathrm{Con}\, L$ satisfy the identity (E_n) and let $p \to q_k$ for all $k \in \{1, \ldots, n-1\}$ and for prime quotients $p = a/b, q_1 = u_1/v_1, \ldots, q_{n-1} = u_{n-1}/v_{n-1}$ of L. We set $q_0 := a/b$ and $\theta_0 := \theta_{a,b},\ \theta_1 := \theta_{u_1,v_1},\ \theta_{n-1} := \theta_{u_{n-1},v_{n-1}},\ \theta_n := \Delta$. We shall be using the abbreviations $q_i \in \theta$ ($q_i \notin \theta$) for $(u_i, v_i) \in \theta$ ($(u_i, v_i) \notin \theta$).

Firstly, by the (RS)-weak modularity of L, for every $i, j \in \{1, \ldots, n-1\}$, $i \neq j$, the conditions $p \to q_i$ and $p \to q_j$ with $q_i \in \theta_i$ and $q_j \in \theta_j$ imply that $q_i \in \theta_j$ or $q_j \in \theta_i$ (we again note that the condition (III) in Definition 3.4 cannot be satisfied). By Lemma 2.1, $q_i \in \theta_j$ ($q_j \in \theta_i$) implies $u_j/v_j \to u_i/v_i$ ($u_i/v_i \to u_j/v_j$). So there exists an ordering $i_1 \leq i_2 \leq \ldots \leq i_{n-1}$ of the set $\{1, \ldots, n-1\}$ such that $q_{i_j} \to q_{i_{j+1}}$ for $j = 1, \ldots, n-2$. W.l.o.g. let us assume that $i_j = j$ for $j \in \{1, \ldots, n-1\}$ so that $q_1 \to q_2, \ldots, q_{n-2} \to q_{n-1}$. Of course, we trivially have $p \to q_0$ and $q_0 \to q_1$.

Secondly, by the (E_n)-modularity of L, the projections $p \to q_k$ with $q_i \in \theta_i$ where $k \in \{0, \ldots, n-1\}$ imply that the condition (I_j) is satisfied (we again note that the condition (II) in Definition 4.4 cannot be satisfied). Hence there is $j \in \{0, \ldots, n-1\}$ with $q_j \in \theta_{j+1}$. The case $j = n-1$ is clearly impossible. If $j = 0$ then we have $(a, b) \in \theta_{u_1,v_1}$, if $j \in \{1, \ldots, n-2\}$ then we have $(u_j, v_j) \in \theta_{u_{j+1},v_{j+1}}$. Now Lemma 2.1 gives us for our prime quotients that $u_1/v_1 \to a/b$ in case $j = 0$ and $u_{j+1}/v_{j+1} \to u_j/v_j$ in case $j \in \{1, \ldots, n-2\}$. Hence $q_1 \to p$ or $q_j \to q_{j+1}$ and $q_{j+1} \to q_j$ for $j \in \{1, \ldots, n-2\}$. We have proven the necessity.

For the converse, assume the given hypothesis, whence, by reordering the given quotients (if necessary), we can assume that for any prime quotients $p, q_1, \ldots, q_{n-1}$ of L, the projections $p \to q_k$ for $k \in \{1, \ldots, n-1\}$ imply that the condition (i) in the hypothesis holds or the following condition (ii)' holds: there exists $j \in \{1, \ldots, n-2\}$ such that there are projections $q_j \to q_{j+1}$ and $q_{j+1} \to q_j$.

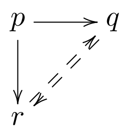
Suppose on the contrary that $\mathrm{Con}\, L$ does not satisfy the identity (E_n), hence there are congruences $\theta_1, \ldots, \theta_{n+1} \in \mathrm{Con}\, L$ and elements $a, b \in L, a > b$, such that $(a, b) \notin (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \ldots \vee (\theta_n * \theta_{n+1})$. This means that for a maximal chain $b = x_0 \leq x_1 \leq \ldots \leq x_m = a$ between the elements a, b (such a chain exists as L is semi-discrete) we have that there is $i \in \{0, \ldots, m-1\}$ such that $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$ for all $j \in \{1, \ldots, n\}$. By Corollary 2.3(ii) this implies that for all $j \in \{1, \ldots, n\}$ there exist projections $x_{i+1}/x_i \to u_j/v_j$

with $(u_j, v_j) \in \theta_j$, $(u_j, v_j) \notin \theta_{j+1}$ for some elements $u_j > v_j$, $j = 1, \dots, n$ in L. Since for every j there is a maximal chain $v_j = y_{j0} \le y_{j1} \le \dots \le y_{jm_j} = u_j$ between the elements u_j, v_j, for every j there is a prime quotient $y_{j,k+1}/y_{jk}$ with $(y_{jk}, y_{j,k+1}) \in \theta_j$ and $(y_{jk}, y_{j,k+1}) \notin \theta_{j+1}$. W.l.o.g. we can assume that the quotients $q_j := u_j/v_j$ are already prime for all $j \in \{1, \dots, n\}$. Of course, the quotient $p := x_{i+1}/x_i$ is prime, too.

Now by our hypothesis we have that for the prime quotients $p, q_2, \dots, q_n$, the projections $p \to q_2, \dots, p \to q_n$ imply one of the conditions (i), (ii). If (i) holds then there is $j \in \{2, \dots, n\}$ such that $q_j \to p$; since $q_j \in \theta_j$, we obtain $p \in \theta_j$ and as $j > 1$ and $p \to q_{j-1}$, we finally obtain $q_{j-1} \in \theta_j$, a contradiction. If (ii) holds then there exists $j \in \{2, \dots, n-1\}$ such that there are projections $q_j \to q_{j+1}$ and $q_{j+1} \to q_j$. As $q_{j+1} \in \theta_{j+1}$, we obtain $q_j \in \theta_{j+1}$, a contradiction. □

Analogously as above, from our Theorem 3.6 one can obtain the characterization of semi-discrete lattices with relative Stone congruence lattices derived in [4, p. 87]:

Corollary 5.2. *Let L be a semi-discrete lattice. Then* $\mathrm{Con}\, L$ *is a relative Stone lattice if and only if for any prime quotients p, q, r of L, the projections $p \to q$ and $p \to r$ imply that $q \to r$ or $r \to q$.*



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Department of Mathematics, Faculty of Natural Sciences, Matej Bel University, Banská Bystrica, Slovakia

E-mail address: dguffova@yahoo.com, mhaviar@fpv.umb.sk

URL: http://www.fpv.umb.sk/~mhaviar