# ON CONGRUENCE LATTICES OF LATTICES

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Abstract. The presented work is related to the problems III.5 and III.6 of G. Grätzer's monograph [2] which ask for a characterization of lattices whose congruence lattice belongs to the $n$-th Lee's equational class $B_n$ of distributive pseudocomplemented lattices for $n \geq 1$. We present self-contained proofs for the characterizations of lattices with relative $(L_n)$- and relative Stone congruence lattices. As corollaries we obtain descriptions of lattices with Stone and $(L_n)$-congruence lattices, as well as descriptions of semi-discrete lattices with relative Stone and relative $(L_n)$-congruence lattices for arbitrary $n \geq 1$.

## 1. Introduction

One of the basic facts about the congruence lattices of lattices is that they are distributive and pseudocomplemented. T. Tanaka [11], P. Crawley [1], G. Grätzer and E. T. Schmidt [3] have characterized those lattices whose congruence lattices are Boolean. In the monograph [2], G. Grätzer posed problems (problems III.5 and III.6) of characterizing those lattices whose congruence lattices considered as pseudocomplemented lattices belong to the $n$th Lee's equational class $B_n$ of distributive pseudocomplemented lattices described by the identity

$$(L_n) \quad (x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1.$$

Distributive pseudocomplemented lattices satisfying the identity $(L_n)$ are called $(L_n)$-lattices [6], [7]. As the class $B_1$ is the class of all Stone lattices, $(L_1)$-lattices are in fact Stone lattices. Lattices whose congruence lattices are Stone have been characterized by T. Katriňák [8]. Later, M. Haviar [6] characterized lattices with $(L_n)$-congruence lattices for arbitrary $n \geq 1$.

Distributive pseudocomplemented lattices in which every interval satisfies the identity $(L_n)$ are called relative $(L_n)$-lattices. In [4], M. Haviar and T. Katriňák characterized lattices with relative Stone congruence lattices. Lattices with relative $(L_n)$-congruence lattices were characterized later by M. Haviar in [6]. Semi-discrete lattices with $(L_n)$- and relative $(L_n)$-congruence lattices were characterized by M. Haviar and T. Katriňák in [7].

The congruence lattices of lattices are also relatively pseudocomplemented, hence they can be investigated as Heyting algebras. It is natural to seek for a characterization of lattices whose congruence lattices satisfy identities formulated in terms of relative pseudocomplement. In particular, relative $(L_n)$-lattices can be characterized by the identity

$$(L'_n) \quad (x_1 \wedge \ldots \wedge x_n) * y \vee (x_1 * y \wedge \ldots \wedge x_n) * y \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n * y) * y = 1.$$

In [7] only semi-discrete lattices whose congruence lattices satisfy the identity $(L'_n)$ were described. In this work we present a description of arbitrary lattices whose congruence lattices considered as Heyting algebras satisfy the identity $(L'_n)$ (section 4). In particular, one obtains a description of lattices with relative $(L_1)$-congruence lattices. In Section 3 we give a slightly different description of lattices with relative Stone congruence lattices than is the one obtained in Section 4 in case $n = 1$.

Our method is alternative to the one presented in [6] and [4] where the identity $(L'_n)$ was not used and the respective descriptions of lattices with relative $(L_n)$- and relative Stone congruence lattices were presented by translating the corresponding conditions for factor lattices $L/\pi$ ($\pi$ is a congruence of $L$) with $(L_n)$- and Stone congruence lattices without the need to write down the proofs for the given characterizations. In our approach presented here we entirely use the identities $(L'_n)$ and we actually write down self-contained proofs for the characterizations of lattices with relative $(L_n)$- and relative Stone congruence lattices.

## 2. Preliminaries

The following basic concepts and facts can be found in [2], [4], [7] or [6].

Let $\operatorname{Con} L$ denote the lattice of all congruences on a lattice L with $\Delta$ and $\nabla$, the smallest and the largest congruence relation. The lattice $\operatorname{Con} L$ is distributive, moreover $\operatorname{Con} L$ satisfies the infinite distributivity law

$$\theta \wedge \bigvee(\alpha_i : i \in I) = \bigvee(\theta \wedge \alpha_i : i \in I)$$

for any $\theta, \alpha_i \in \operatorname{Con} L$.

It follows that for any $\alpha, \beta \in \operatorname{Con} L$ there exists a largest congruence $\delta$ such that $\alpha \wedge \delta \leq \beta$. It is obvious that $\delta = \bigvee(\sigma : \alpha \wedge \sigma \leq \beta)$. The congruence $\delta$ is called the *relative pseudocomplement of* $\alpha$ *with respect to* $\beta$ and denoted by $\alpha * \beta$. Therefore $\langle \operatorname{Con} L, \vee, \wedge, *, \Delta, \nabla \rangle$ is a complete relatively pseudocomplemented lattice, i.e. a complete Heyting algebra.

Recall that an algebra $\langle H, \vee, \wedge, *, 0, 1 \rangle$ of type (2, 2, 2, 0, 0) is a *Heyting algebra* if it satisfies:

(H1) $\langle H, \vee, \wedge \rangle$ is a distributive lattice,
(H2) $x \wedge 0 = 0$, $x \vee 1 = 1$,
(H3) $x * x = 1$,
(H4) $(x * y) \wedge y = y$, $x \wedge (x * y) = x \wedge y$,
(H5) $x * (y \wedge z) = (x * y) \wedge (x * z)$, $(x \vee y) * z = (x * z) \wedge (y * z)$.

The Heyting algebras were introduced by G. Birkhoff under the name *Brouwerian algebras.*

K. B. Lee [10] has shown that the lattice of all equational subclasses of the class $B_\omega$ of all distributive pseudocomplemented lattices (*p-algebras*) is a chain

$$B_{-1} \subset B_0 \subset B_1 \subset \ldots \subset B_n \subset \ldots \subset B_\omega$$

of type $\omega+1$, where $B_{-1}, B_0, B_1$ are the classes of all trivial p-algebras, Boolean algebras, and Stone algebras, respectively. Moreover, a distributive pseudocomplemented lattice belongs to the class $B_n$ $(n \geq 1)$ if and only if it satisfies the identity

$$(L_n) \quad (x_1 \wedge \ldots \wedge x_n)^* \vee (x_1^* \wedge \ldots \wedge x_n)^* \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n^*)^* = 1,$$

i.e. is an $(L_n)$-lattice.

A distributive relatively pseudocomplemented lattice $\langle L, \vee, \wedge, *, 0, 1\rangle$ is a relative Stone lattice if and only if

$$x * y \vee (x * y) * y = 1$$

for every $x, y \in L$. A distributive relatively pseudocomplemented lattice $L$ is a relative $(L_n)$-lattice $(n \geq 1)$ if and only if it satisfies the identity

$$(L_n') \quad (x_1 \wedge \ldots \wedge x_n) * y \vee (x_1 * y \wedge \ldots \wedge x_n) * y \vee \ldots \vee (x_1 \wedge \ldots \wedge x_n * y) * y = 1.$$

One of the mostly used concepts in this work is the concept of weak projectivity of quotients. We denote $a/b$ an ordered pair of elements a, b of a lattice $L$ satisfying $b \leq a$; $a/b$ is called a *quotient* of L. A quotient $c/d$ is a *subquotient* of $a/b$ if $b \leq d \leq c \leq a$. We call a/b a *proper quotient* if $b < a$. If $b \prec a$, i.e. $b$ is covered by $a$, then a/b is called a *prime quotient.*

We will say that a quotient $a/b$ is *weakly projective* to a quotient $c/d$ and use the notation $a/b \to c/d$ if there exist finitely many elements $x_1, \ldots, x_n \in L$ such that

$$c = (\ldots((a \vee x_1) \wedge x_2) \vee \ldots) \vee x_n,$$

$$d = (\ldots((b \vee x_1) \wedge x_2) \vee \ldots) \vee x_n.$$

The importance of weak projectivity in the description of lattice congruences is given by the following two lemmas.

**Lemma 2.1.** ([6], Lemma 1) *For any principal congruence* $\theta_{a,b} \in \operatorname{Con} L$,

$$(c, d) \in \theta_{a,b},$$

$(d \leq c, b \leq a)$ *if and only if there is a finite chain* $d = y_0 \leq \ldots \leq y_n = c$ *such that* $a/b \to y_{i+1}/y_i$ *for all* $i \in \{0, \ldots, n-1\}$.

**Lemma 2.2.** ([6], Lemma 2) *Let $L$ be a lattice and $\theta, \varphi \in \operatorname{Con} L$. then the relative pseudocomplement of $\theta$ with respect to $\varphi$ is*

$$\theta * \varphi = \bigvee(\theta_{u,v}, (u,v) \in S),$$

*where $S$ is the set of all pairs of elements $(u,v)$ $(u, v \in L)$ such that $u/v \to z/t$ and $(z,t) \in \theta$ implies $(z,t) \in \varphi$ for all $z, t \in L$.*

## 3. Lattices with relative Stone congruence lattices

In this section we give a description of lattices with relative Stone congruence lattices.

**Definition 3.1.** ([4], Definition 1) *Let $L$ be a lattice, $\pi \in \operatorname{Con} L$ and $a/b, u/v$ quotients of $L$. Then $L$ is said to be $\pi$-**almost weakly modular** whenever $a/b \to u/v$ and $(u,v) \notin \pi$ imply the existence of a subquotient $a_1/b_1 \subseteq a/b$ with $(a_1, b_1) \notin \pi$ such that for every quotient $r/s$ with $a_1/b_1 \to r/s$ and $(r,s) \notin \pi$ there exists a quotient $z/t$ with $r/s \to z/t$ , $u/v \to z/t$ and $(z,t) \notin \pi$.*

**Definition 3.2.** ([4], Definition 2) *Let $L$ be a lattice and $\theta, \pi \in \operatorname{Con} L$, $\theta \geq \pi$. Then $\theta$ is said to be $\pi$-**weakly separable** if for any $a < b$ in $L$ there exists a chain $a = z_0 \leq z_1 \leq \ldots \leq z_n = b$ such that for each $i \in \{0, \ldots, n-1\}$ either*

(i) *$z_{i+1}/z_i \to u/v$ and $(u,v) \in \theta$ imply $(u,v) \in \pi$ or*
(ii) *for every subquotient $r/s \subseteq z_{i+1}/z_i$ with $(r,s) \notin \pi$, there exists a quotient $u/v$ with $r/s \to u/v$ and $(u,v) \in \theta$, $(u,v) \notin \pi$.*

**Theorem 3.3.** ([4], Theorem 2) *Let $L$ be a lattice. The lattice $\operatorname{Con} L$ is relative Stone if and only if for every $\pi \in \operatorname{Con} L$ the following conditions hold:*

(1) *$L$ is $\pi$-almost weakly modular and*
(2) *every congruence $\theta \geq \pi$ is $\pi$-weakly separable.*

*Proof.* First we will prove the necessity.

Let $\operatorname{Con} L$ be relatively Stone lattice, i. e. it satisfies the identity

$$(\theta * \pi) \vee ((\theta * \pi) * \pi) = \nabla,$$

for all congruences $\theta, \pi \in \operatorname{Con} L$. Let $a/b, u/v$ be quotients of $L$ such that $a/b \to u/v$ with $(u,v) \notin \pi$ and $a > b, u > v$. Set

$$\phi := \theta_{u,v} \vee \pi.$$

Since $\operatorname{Con} L$ is relatively Stone, it follows that

$$(a,b) \in (\phi * \pi) \vee ((\phi * \pi) * \pi) = (\theta_{u,v} * \pi) \vee ((\theta_{u,v} * \pi) * \pi),$$

so there exists a chain $b = c_0 \leq c_1 \leq \ldots \leq c_n = a$ such that for every $i \in \{0, \ldots, n-1\}$

$$(c_{i+1}, c_i) \in (\theta_{u,v} * \pi) \text{ or } (c_{i+1}, c_i) \in ((\theta_{u,v} * \pi) * \pi).$$

If for every $i \in \{0, \ldots, n-1\}$ the first case holds, we get $(a, b) \in \theta_{u,v} * \pi$, that is, $(u, v) \in \theta_{u,v} * \pi$, so $(u, v) \in \pi$, a contradiction.

Thus there is a subquotient $a_1/b_1 \subseteq a/b$ such that $(a_1, b_1) \notin (\theta_{u,v} * \pi)$ and $(a_1, b_1) \in ((\theta_{u,v} * \pi) * \pi)$. Let $r/s$ be a quotient such that $a_1/b_1 \to r/s$ and $(r, s) \notin \pi$. Then $(r, s) \in ((\theta_{u,v} * \pi) * \pi)$.

Whenever the conditions $r/s \to z'/t'$, $(z', t') \in \theta_{u,v}$ would imply $(z', t') \in \pi$ we would get $(r, s) \in (\theta_{u,v} * \pi)$, so $(r, s) \in \pi$ would also hold, a contradiction.

Hence there exists a quotient $z'/t'$ such that $r/s \to z'/t'$, with $(z', t') \in \theta_{u,v}$ and $(z', t') \notin \pi$. Since $(z', t') \in \theta_{u,v}$, there exists a subquotient $z/t \subseteq z'/t'$ such that $u/v \to z/t$. As also $r/s \to z/t$, $\mathrm{Con}\, L$ is $\pi$-almost weakly modular.

Now let $\theta \in \mathrm{Con}\, L$ with $\theta \geq \pi$. Since $\mathrm{Con}\, L$ is relative Stone lattice, $(a, b) \in (\theta * \pi) \vee ((\theta * \pi) * \pi)$ for any $a > b$. Therefore there exists a chain $b = z_0 \leq \ldots \leq z_m = a$ such that

$$(z_{i+1},\ z_i) \in (\theta * \pi) \text{ or } (z_{i+1},\ z_i) \in ((\theta * \pi) * \pi).$$

In the first case we get that $z_{i+1}/z_i \to u/v$ and $(u, v) \in \theta$ implies $(u, v) \in \pi$. So we get the condition (i) from the Definition 3.2. Now let $(z_{i+1},\ z_i) \in ((\theta*\pi)*\pi)$. Let $r/s$ be a proper subquotient of the quotient $z_{i+1}/z_i$ with $(r, s) \notin \pi$ If for every $u \geq v$ the conditions $r/s \to u/v$ and $(u, v) \in \theta$ imply $(u, v) \in \pi$, then $(r, s) \in (\theta * \pi)$. So we would get $(r, s) \in ((\theta * \pi) * \pi)$ and $(r, s) \in (\theta * \pi)$, which yields $(r, s) \in \pi$, a contradiction.

So there exists a quotient $u/v$ such that $r/s \to u/v$, $(u, v) \in \theta$ and $(u, v) \notin \pi$. The $\pi$-weakly separability of any congruence $\theta \in \mathrm{Con}\, L$ has been proved.

Now we will prove the sufficiency. Let $a < b$ and let $\theta, \pi \in \mathrm{Con}\, L$, $\theta \geq \pi$. From $\pi$-weakly separability of congruence $\theta$ follows the existence of a chain $a = z_0 \leq \ldots \leq z_n = b$ such that for every $i \in \{0, \ldots, n-1\}$ (i) or (ii) from Definition 3.2 holds. If (i) holds, we get $(z_i, z_{i+1}) \in (\theta * \pi)$. Now let assume that (i) from the Definiton 3.2 does not hold and that (ii) from the Definition 3.2 holds for $(z_i, z_{i+1})$. We will distinguish two cases:

I. Let assume that $z_i/z_{i+1} \to u/v$, $(u, v) \in (\theta * \pi)$ imply $(u, v) \in \pi$. We get $(z_i, z_{i+1}) \in ((\theta * \pi) * \pi)$.

II. There remains the case when $z_i/z_{i+1} \to u/v$, $(u, v) \in (\theta*\pi)$ but $(u, v) \notin \pi$. The $\pi$-almost weakly modularity of $\mathrm{Con}\, L$ yields the existence of a subquotient $a_1/b_1 \subseteq z_i/z_{i+1}$, with $(a_1, b_1) \notin \pi$, such that for every quotient $r/s$ with $a_1/b_1 \to r/s$ and $(r, s) \notin \pi$ there exists a quotient $z/t$ with $u/v \to z/t$ and $r/s \to z/t$ with $(z, t) \notin \pi$. From (ii) of Definiton 3.2 it follows that there exists a quotient $u/v$ such that $a_1/b_1 \to u/v$, $(u, v) \in \theta$ and $(u, v) \notin \pi$. By $\pi$-almost weakly modularity of $L$ there exists a quotient $z/t$ such that $u/v \to z/t$ and $(z, t) \notin \pi$. Then $(z, t) \in \theta$ and $(z, t) \in \theta * \pi$, so $(z, t) \in \pi$, a contradiction.

Therefore the case II. cannot occur, so for every $i \in \{0, \dots, n-1\}$

$$(z_{i+1}, z_i) \in (\theta * \pi) \text{ or } (z_{i+1},\ z_i) \in ((\theta * \pi) * \pi)$$

holds. Hence $\mathrm{Con}\, L$ is relative Stone lattice. □

Theorem 3.3 yields the following statements.

**Corollary 3.4.** ([4], Theorem 1) *Let $L$ be a lattice. Then $\mathrm{Con}\, L$ is a Stone lattice if and only if the following conditions hold:*

(1) *$L$ is $\Delta$-almost weakly modular and*
(2) *every congruence of $L$ is $\Delta$-weakly separable.*

**Corollary 3.5.** ([4], Corollary to Theorem 5) *Let $L$ be a semi-discrete lattice. Then $\mathrm{Con}\, L$ is a relative Stone lattice if and only if for any prime quotients $p, q, r$ of $L$ satisfying $p \to q$ and $p \to r$ either $q \to r$ or $r \to q$ holds.*

Note that a lattice $L$ is called *semi-discrete* if between all comparable pairs of elements of $L$ there exists a finite maximal chain.

## 4. Lattices with relative $L_n$-congruence lattices

In this section we give we a description of arbitrary lattices whose congruence lattices considered as Heyting algebras satisfy the identity $(L_n')$.

**Definition 4.1.** ([5], Definition 3) *Let $L$ be a lattice, $a/b, u_1, /v_1, \dots, u_{n+1}/v_{n+1}$ be nontrivial quotients of $L$ and $n \geq 1$. Then $L$ is said to be $(\pi - n)$-**weakly modular** whenever*

$$a/b \to u_i/v_i \text{ and } (u_i, v_i) \notin \pi,\ i = 1, \dots, n+1$$

*imply that one of the following conditions holds:*

(i) *there exist $i, j \in \{1, \dots, n+1\}, i \neq j$ and a quotient $u/v$ such that $u_i/v_i \to u/v$, $u_j/v_j \to u/v$ with $(u, v) \notin \pi$.*
(ii) *for all $i \in \{1, \dots, n+1\}$ there is a proper subquotient $r_i/s_i \subset a/b$ such that $(r_i, s_i) \notin \pi$ and $(r_i, a) \notin \pi$ or $(s_i, b) \notin \pi$ and a quotient $z_i/t_i$, such that $r_i/s_i \to z_i/t_i$, $u_i/v_i \to z_i/t_i$ and $(z_i, t_i) \notin \pi$.*

**Definition 4.2.** ([5], Definition 4) *Let $L$ be a lattice, $\pi \in \mathrm{Con}\, L$ and $n \geq 1$. Then an (unordered) $n$-tuple $\theta_1, \dots, \theta_n$ ($\theta_1, \dots, \theta_n \geq \pi$) is said to be $(\pi - n)$-**separable** if for any $b < a$ there exists a chain $b = z_0 \leq z_1 \leq \dots \leq z_m = a$ such that for every $i \in \{0, \dots, m-1\}$ either*

(i) *$z_{i+1}/z_i \to u/v$ and $(u, v) \in (\theta_1 \cap \dots \cap \theta_n)$ imply $(u, v) \in \pi$ or*
(ii) *there exists some $j \in \{1, \dots, n\}$ such that for every proper subquotient $r/s \subset z_{i+1}/z_i$ with $(r, s) \notin \pi$ and $(r, z_{i+1}) \notin \pi$ or $(s, z_i) \notin \pi$ the following holds: $r/s \to u/v$, $(u, v) \in (\theta_1 \cap \dots \cap \theta_{j-1} \cap \theta_{j+1} \cap \dots \cap \theta_n)$ and*

*$(u,v) \notin \pi$ imply the existence of a quotient $u'/v'$ such that $u/v \to u'/v'$ and $(u',v') \in \theta_j$, $(u',v') \notin \pi$.*

**Theorem 4.3.** ([5], Theorem 4) *Let $L$ be a lattice and $n \geq 1$.* $\operatorname{Con} L$ *is relative $(L_n)$-lattice if and only if for every $\pi \in \operatorname{Con} L$ the following conditions hold:*

(i) *$L$ is $(\pi - n)$-weakly modular and*

(ii) *every $n$-tuple of congruences $\theta_1, \ldots, \theta_n$ on $L$ such that $\theta_i \geq \pi$ for all $i = 1, \ldots, n$ is $(\pi - n)$-separable.*

*Proof.* Assume that $\operatorname{Con} L$ satisfies the identity

$$(L_n') \ ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi) \vee ((\theta_1 * \pi \wedge \ldots \wedge \theta_n) * \pi) \vee \ldots \vee ((\theta_1 \wedge \ldots \wedge \theta_n * \pi) * \pi) = \nabla.$$

We shall prove that $L$ is $(\pi - n)$-weakly modular. Let $\pi \in \operatorname{Con} L$ and let $a/b, u_1/v_1, \ldots, u_{n+1}/v_{n+1}$ be nontrivial quotients in L such that $a/b \to u_i/v_i$ and $(u_i, v_i) \notin \pi$ for $i = 1, \ldots, n+1$. Consider there are no $i, j \in \{1, \ldots, n+1\}$, $i \neq j$ and a quotient $u/v$, $(u,v) \notin \pi$ such that $u_i/v_i \to u/v, u_j/v_j \to u/v$. Set

$$\phi_1 := \theta_{u_1,v_1} \vee \pi, \ \ldots, \ \phi_{n+1} := \theta_{u_{n+1},v_{n+1}} \vee \pi.$$

We shall prove that

$$(1) \qquad (\phi_1 * \pi) \vee \ldots \vee (\phi_{n+1} * \pi) = \nabla.$$

We will show that $\phi_i \cap \phi_j = \pi$ for all $i, j \in \{1, \ldots, n+1\}, i \neq j$. It is obvious that $\pi \subseteq \phi_i \cap \phi_j$. To prove the equality suppose the existence of elements $u, v \in L$, $u > v$, such that $(u,v) \notin \pi$ and $(u,v) \in (\theta_i \cap \theta_j)$. By distributivity we get $\phi_i \cap \phi_j = (\theta_{u_i,v_i} \wedge \theta_{u_j,v_j}) \vee \pi$. Thus there exists a chain $v = c_0 \leq \ldots \leq c_n = u$ such that $(c_{k+1}, c_k) \in (\theta_{u_i,v_i} \wedge \theta_{u_j,v_j})$ or $(c_{k+1}, c_k) \in \pi$. Since $(u,v) \notin \pi$ there exists a nontrivial subquotient $u'/v' \subseteq u/v$ such that $(u',v') \in (\theta_{u_i,v_i} \wedge \theta_{u_j,v_j})$ and $(u',v') \notin \pi$. By Lemma 1 there exists a nontrivial subquotient $u''/v'' \subseteq u'/v'$ such that $u_i/v_i \to u''/v''$, $u_j/v_j \to u''/v''$ and $(u'',v'') \notin \pi$, a contradiction. Hence $\phi_i \cap \phi_j = \pi$ and $\phi_i \leq \phi_j * \pi$ for all $i, j \in \{1, \ldots, n+1\}, i \neq j$.

In the case $n = 1$ we have $(\phi_1 * \pi) \vee ((\phi_1 * \pi) * \pi) = \nabla$. Since $\phi_2 \leq \phi_1 * \pi$, we get $\phi_2 * \pi \geq (\phi_1 * \pi) * \pi$. So

$$\nabla = (\phi_1 * \pi) \vee ((\phi_1 * \pi) * \pi) \leq (\phi_1 * \pi) \vee (\phi_2 * \pi),$$

thus (1) holds. Now assume $n \geq 2$. Set

$$\alpha_1 := \phi_2 \vee \phi_3 \vee \ldots \vee \phi_n \vee \phi_{n+1}$$

$$\alpha_2 := \phi_1 \vee \phi_3 \vee \ldots \vee \phi_n \vee \phi_{n+1}$$

$$\vdots$$

$$\alpha_n := \phi_1 \vee \phi_2 \vee \ldots \vee \phi_{n-1} \vee \phi_{n+1}.$$

We have

$$((\alpha_1 \wedge \ldots \wedge \alpha_n) * \pi) \vee ((\alpha_1 * \pi \wedge \ldots \wedge \alpha_n) * \pi) \vee \ldots \vee ((\alpha_1 \wedge \ldots \wedge \alpha_n * \pi) * \pi) = \nabla.$$

We will prove that

(2) $(\alpha_1 \wedge \ldots \wedge \alpha_n) = \phi_{n+1},\ (\alpha_1 * \pi \wedge \ldots \wedge \alpha_n) = \phi_1, \ldots,\ (\alpha_1 \wedge \ldots \wedge \alpha_n * \pi) = \phi_n.$

First we will show that

$$\alpha_1 \wedge \ldots \wedge \alpha_n = \phi_{n+1}.$$

Clearly $\phi_{n+1} \subseteq \alpha_1 \wedge \ldots \wedge \alpha_n$. Suppose on the contrary that there exist $u, v \in L$, $(u, v) \in (\alpha_1 \wedge \ldots \wedge \alpha_n)$ and $(u, v) \notin \phi_{n+1}$. As $(u, v) \in \alpha_1$, there exists some $i \in \{2, \ldots, n\}$ and a subquotient $u'/v' \subseteq u/v$, $(u', v') \notin \pi$ such that $(u', v') \in \phi_i$ and $(u', v') \notin \phi_{n+1}$. We also have $(u', v') \in \alpha_i$, so there exist $j \in \{1, \ldots, n\} - \{i\}$ and a subquotient $u''/v'' \subseteq u'/v'$, $(u'', v'') \notin \pi$ such that $(u'', v'') \in \phi_j$. Then $(u'', v'') \in (\phi_i \cap \phi_j)$ that contradicts $\phi_j \cap \phi_i = \pi$, for $i \neq j$. Therefore

$$\alpha_1 \wedge \ldots \wedge \alpha_n = \phi_{n+1}.$$

Also

$$\alpha_i * \pi = (\phi_1 * \pi) \wedge \ldots \wedge (\phi_{i-1} * \pi) \wedge (\phi_{i+1} * \pi) \wedge \ldots \wedge (\phi_{n+1} * \pi).$$

Using the fact that $\phi_i \cap \phi_j = \pi$, for all $i \neq j$ and the distributivity law we get

$$\alpha_1 \wedge \ldots \wedge (\alpha_i * \pi) \wedge \ldots \wedge \alpha_n = ((\phi_1 * \pi) \wedge \ldots \wedge (\phi_{i-1} * \pi) \wedge \phi_i \wedge (\phi_{i+1} * \pi) \wedge \ldots \wedge (\phi_n * \pi)) \vee \pi.$$

As $\phi_i \leq \phi_j * \pi$ and $\pi \leq \phi_i$, we have

$$\alpha_1 \wedge \ldots \wedge (\alpha_i * \pi) \wedge \ldots \wedge \alpha_n = \phi_i \vee \pi = \phi_i$$

for $i = 1, \ldots, n$. Thus the equalities in (2) hold. Now (1) follows from the assumption and (2). So, $(a, b) \in (\phi_1 * \pi) \vee \ldots \vee (\phi_{n+1} * \pi)$.

Let consider the existence of $i \in \{1, \ldots, n+1\}$ with $(a, b) \in (\phi_i * \pi)$. Then also $(u_i, v_i) \in \phi_i \cap (\phi_i * \pi)$, so we get $(u_i, v_i) \in \pi$, a contradiction. Thus for every $i \in \{1, \ldots, n+1\}$ there is a nontrivial proper subquotient $r_i/s_i \subset a/b$, where $(r_i, a) \notin \pi$ or $(s_i, b) \notin \pi$ and $(r_i, s_i) \notin \pi$ such that $(r_i, s_i) \notin (\phi_i * \pi)$. Then for every $i \in \{1, \ldots, n+1\}$ there is a quotient $z_i'/t_i'$ with $r_i/s_i \to z_i'/t_i'$ and $(z_i', t_i') \in \phi_i$ and $(z_i', t_i') \notin \pi$. Thus for every $i \in \{1, \ldots, n+1\}$ there is a proper subquotient $r_i/s_i \subset a/b$, where $(r_i, a) \notin \pi$ or $(s_i, b) \notin \pi$ and $(r_i, s_i) \notin \pi$ and a quotient $z_i/t_i$, $(z_i, t_i) \notin \pi$ such that $r_i/s_i \to z_i/t_i$ and $u_i/v_i \to z_i/t_i$. Hence, L is $(\pi - n)$-weakly modular.

Now, let $\pi \in \operatorname{Con} L$, $\theta_1, \ldots, \theta_n$, $\theta_i \geq \pi$ for $i = 1, \ldots, n$ and $b < a$. Since

$$(a, b) \in ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi) \vee ((\theta_1 * \pi \wedge \ldots \wedge \theta_n) * \pi) \vee \ldots \vee ((\theta_1 \wedge \ldots \wedge \theta_n * \pi) * \pi),$$

there is a chain $b = z_0 \leq \ldots \leq z_m = a$ such that for all $i = 1, \ldots, m-1$

$$(z_{i+1}, z_i) \in ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi) \text{ or}$$

$$(z_{i+1}, z_i) \in (\theta_1 \wedge \ldots \wedge (\theta_j * \pi) \wedge \ldots \wedge \theta_n) * \pi \text{ for some } j \in \{1, \ldots, n\}.$$

In the first case we get (i) from the definition of $(\pi - n)$-separability.

We should show that in the other case the condition (ii) from the definition 4.2 holds. Let $(z_{i+1}, z_i) \in (\theta_1 \wedge \ldots \wedge (\theta_j * \pi) \wedge \ldots \wedge \theta_n) * \pi$ for some $j \in \{1, \ldots, n\}$. Further let $r/s \subset z_{i+1}/z_i$ be a nontrivial proper subquotient, $(r, s) \notin \pi$ and $(r, z_{i+1}) \notin \pi$ or $(s, z_i) \notin \pi$, and let $r/s \to u/v$ such that $(u, v) \notin \pi$ and $(u, v) \in (\theta_1 \wedge \ldots \wedge \theta_{j-1} \wedge \theta_{j+1} \wedge \ldots \wedge \theta_n)$.

Suppose that for any $u' \geq v'$, the conditions $u/v \to u'/v'$ and $(u', v') \in \theta_j$ imply $(u', v') \in \pi$. By Lemma 2.2 we obtain $(u, v) \in (\theta_j * \pi)$, hence we get $(u, v) \in (\theta_1 \wedge \ldots \wedge \theta_{j-1} \wedge \theta_j * \pi \wedge \theta_{j+1} \wedge \ldots \wedge \theta_n)$. Since we also have $(u, v) \in ((\theta_1 \wedge \ldots \wedge \theta_{j-1} \wedge \theta_j * \pi \wedge \theta_{j+1} \wedge \ldots \wedge \theta_n) * \pi)$, we get $(u, v) \in \pi$, a contradiction. Therefore there exist elements $u' > v'$ such that $u/v \to u'/v'$ and $(u', v') \in \theta_j$. This yields that every (unordered) n-tuple $\theta_1, \ldots, \theta_n \in \operatorname{Con} L$, $\theta_i \geq \pi$, is $(\pi - n)$-separable.

Conversely, let $L$ be $(\pi - n)$-weakly modular lattice and let every n-tuple $\theta_1, \ldots, \theta_n \in \operatorname{Con} L$, $\theta_i \geq \pi$, be $(\pi - n)$-separable. To prove that $L$ satisfies the identity $(L_n)$ it is sufficient to show that for any $b < a$

$$(a, b) \in ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi \vee (\theta_1 * \pi \wedge \ldots \wedge \theta_n) * \pi \vee \ldots \vee (\theta_1 \wedge \ldots \wedge \theta_n * \pi) * \pi).$$

Let $b < a$. By $(\pi - n)$-weakly separability of $\theta_1, \ldots, \theta_n$, $\theta_i \geq \pi$, there exists a chain $b = c_0 \leq \ldots \leq c_m = a$ such that for all $i = 0, \ldots, m-1$ either the condition (i) or the condition (ii) from the definition 4.2 holds.

In the first case we immediately obtain $(c_{i+1}, c_i) \in ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi)$. Now assume that (i) of 4.2 does not hold, so there is a quotient $u_{n+1}/v_{n+1}$, $(u_{n+1}, v_{n+1}) \notin \pi$ such that
$c_{i+1}/c_i \to u_{n+1}/v_{n+1}$ and also $(u_{n+1}, v_{n+1}) \in (\theta_1 \wedge \ldots \wedge \theta_n)$ and the condition (ii) holds. Two cases can occur:

I. there exists $j \in \{1, \ldots, n\}$ such that the conditions $c_{i+1}/c_i \to u/v$ and $(u, v) \in (\theta_1 \wedge \ldots \wedge (\theta_j * \pi) \wedge \ldots \wedge \theta_n)$ imply $(u, v) \in \pi$. By Lemma 2.2 we get $(c_{i+1}, c_i) \in ((\theta_1 \wedge \ldots \wedge \theta_j * \pi \wedge \ldots \wedge \theta_n) * \pi)$.

II. for every $j \in \{1, \ldots, n\}$ there exists a nontrivial quotient $u_j/v_j$ such that $c_{i+1}/c_i \to u_j/v_j$ and $(u_j, v_j) \in (\theta_1 \wedge \ldots \wedge (\theta_j * \pi) \wedge \ldots \wedge \theta_n)$ with $(u_j, v_j) \notin \pi$. As $L$ is $(\pi - n)$-weakly modular lattice, (i) or (ii) from the Definition 4.1 holds for the quotients $c_{i+1}/c_i$, $u_j/v_j$, $j = 1, \ldots, n+1$. If the condition (i) holds, then there exist $i, j \in \{1, \ldots, n+1\}, i < j$ and a quotient $u/v$ such that $u_i/v_i \to u/v$, $u_j/v_j \to u/v$ with $(u, v) \notin \pi$. But then $(u, v) \in (\theta_i * \pi)$ and $(u, v) \in \theta_i$ whence $(u, v) \in \pi$, a contradiction. Now let the condition (ii) of 4.1 hold. Hence for every $j \in \{1, \ldots, n+1\}$ there exists a proper subquotient $r_j/s_j \subset c_{i+1}/c_i$, $(r_j, s_j) \notin \pi$ and $(r_j, c_{i+1}) \notin \pi$ or $(s_j, c_i) \notin \pi$, and a quotient $z_j/t_j$ such that $r_j/s_j \to z_j/t_j$, $u_j/v_j \to z_j/t_j$ and $(z_j, t_j) \notin \pi$. Thus for all $j = 1, \ldots, n$ we get $(z_j, t_j) \in (\theta_1 \wedge \ldots \wedge (\theta_j * \pi) \wedge \ldots \wedge \theta_n)$. Since the condition (ii) from the Definition 4.2 holds, it follows that for some $j \in \{1, \ldots, n\}$ there exists a quotient $z/t$,

$(z,t) \notin \pi$ with $z_j/t_j \rightarrow z/t$ and $(z,t) \in \theta_j$. Since $(z_j, t_j) \in (\theta_j * \pi)$, we get $(z,t) \in (\theta_j \wedge (\theta_j * \pi))$, so $(z,t) \in \pi$, a contradiction. Therefore the case II is impossible.

So for every $i \in \{1, \ldots, m-1\}$

$$(c_{i+1},\ c_i) \in ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi) \text{ or}$$

$$(c_{i+1},\ c_i) \in ((\theta_1 \wedge \ldots \wedge \theta_j * \pi \wedge \ldots \wedge \theta_n) * \pi) \text{ for some } j \in \{1, \ldots, n\},$$

which yields

$$(a,b) \in ((\theta_1 \wedge \ldots \wedge \theta_n) * \pi) \vee ((\theta_1 * \pi \wedge \ldots \wedge \theta_n) * \pi) \vee \ldots \vee ((\theta_1 \wedge \ldots \wedge \theta_n * \pi) * \pi),$$

so the lattice $L$ satisfies the identity $(L'_n)$. □

As corollaries we obtain the following results.

**Corollary 4.4.** ([6], Theorem 1) *Let $L$ be a lattice and $n \geq 1$.* $\operatorname{Con} L$ *is $(L_n)$-lattice if and only if the following conditions hold:*

(i) *$L$ is $(\Delta - n)$-weakly modular and*

(ii) *every $n$-tuple $\theta_1, \ldots, \theta_n$ from* $\operatorname{Con} L$ *is $(\Delta - n)$-separable.*

**Corollary 4.5.** ([7], Corollary 3) *Let $L$ be a semi-discrete lattice and $n \geq 1$. Then* $\operatorname{Con} L$ *is a relative $(L_n)$-lattice if and only if for any prime quotients $p, q_1, \ldots, q_{n+1}$ of $L$ the relations $p \rightarrow q_k$, $k = 1, \ldots, n+1$ imply $q_i \rightarrow q_j$ or $q_j \rightarrow q_i$ for some $i, j \in \{1, \ldots, n+1\}$, $i \neq j$.*

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