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**Zväzy s relatívne Stoneovými zväzmi  
kongruencií zväzov**

(ŠVOČ)

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**Lattices with relative Stone congruence  
lattices**

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## 1. INTRODUCTION

A well-known fact about the congruence lattices of lattices is that they are distributive and (relatively) pseudocomplemented, hence they can be investigated as Heyting algebras. It is natural to characterise lattices whose congruence lattices satisfy identities formulated in terms of (relative) pseudocomplement.

In [3], G. Grätzer and E. T. Schmidt characterised those lattices whose congruence lattices are Boolean thereby answering G. Birkhoff's problem [1] (cf. [1, Problem 39]). Their result has been presented in terms of weak projectivity of quotients of the lattice. In Theorem 4.3 we generalise Grätzer-Schmidt's result within the subvarieties of relative Stone Heyting algebras which are characterised by the identity

$$(E_n) \quad (x_1 * x_2) \vee (x_2 * x_3) \vee \dots \vee (x_n * x_{n+1}) = 1.$$

As the identity  $(E_2)$  characterises the variety of Boolean algebras, as a corollary we obtain a new characterization of lattices with Boolean congruence lattices (Corollary 4.4).

This is a continuation of the previous papers [9], [5], [7] and [6] by T. Katriňák and M. Haviar. We also give a new characterization, alternative to that in [5], of lattices with relative Stone congruence lattices (Theorem 3.3), using the identity

$$(RS) \quad x * y \vee y * x = 1,$$

which characterises the relative Stone lattices within the variety of Heyting algebras. Finally, we characterise the semi-discrete lattices with relative Stone congruence lattices.

## 2. PRELIMINARIES

Let  $\text{Con } L$  denote the lattice of all congruence relations on a lattice  $L$  with  $\Delta$  and  $\nabla$ , the smallest and the largest congruence relation, respectively. The lattice  $\text{Con } L$  is distributive, moreover  $\text{Con } L$  satisfies the infinite distributivity law (cf. [2])

$$\theta \wedge \bigvee (\alpha_i : i \in I) = \bigvee (\theta \wedge \alpha_i : i \in I)$$

for any  $\theta, \alpha_i \in \text{Con } L$ . It follows that for any pair of congruences  $\alpha, \beta \in \text{Con } L$  there exists a largest congruence  $\delta \in \text{Con } L$  such that  $\alpha \wedge \delta \leq \beta$ . It is clear that  $\delta = \bigvee (\sigma : \alpha \wedge \sigma \leq \beta)$ . The congruence  $\delta$  is called the *relative pseudocomplement of  $\alpha$  with respect to  $\beta$*  and is denoted by  $\alpha * \beta$ . Thus  $\langle \text{Con } L, \vee, \wedge, *, \Delta, \nabla \rangle$  is a complete relatively pseudocomplemented lattice, that is, a complete Heyting algebra.

Let us recall that an algebra  $\langle H, \vee, \wedge, *, 0, 1 \rangle$  of type  $(2, 2, 2, 0, 0)$  is a *Heyting algebra* if it satisfies, for all  $x, y, z \in H$ :

- (H1)  $\langle H, \vee, \wedge \rangle$  is a distributive lattice,
- (H2)  $x \wedge 0 = 0$ ,
- (H3)  $x * x = 1$ ,
- (H4)  $(x * y) \wedge y = y$ ,  $x \wedge (x * y) = x \wedge y$ ,
- (H5)  $x * (y \wedge z) = (x * y) \wedge (x * z)$ ,  $(x \vee y) * z = (x * z) \wedge (y * z)$ .

Within the variety of bounded distributive lattices, the *bounded relative Stone lattices*, that is those in which all intervals are Stone lattices, are characterised exactly as Heyting algebras satisfying the identity  $(RS) x*y \vee y*x = 1$  (see [10, 2.9. and 2.10]). In [8] it was shown that the lattice of all subvarieties of the variety of bounded relative Stone lattices is isomorphic to the chain of type  $\omega + 1$  and that a Heyting algebra  $L$  belongs to the  $n$ -th ( $n \geq 2$ ) subvariety if and only if it satisfies the identity

$$(E_n) \quad (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \dots \vee (\theta_n * \theta_{n+1}) = \nabla.$$

The subvariety satisfying  $(E_2)$  is exactly the variety of all Boolean algebras.

One of the mostly used concepts in this work is the concept of weak projectivity of quotients. We denote by  $a/b \rightarrow c/d$  the weak projectivity of quotients of the lattice  $L$ . We call a quotient  $a/b$  *non-trivial* if  $b < a$  in  $L$ . A *proper* subquotient  $a'/b' \subset a/b$  is a quotient  $a'/b'$  such that  $a \leq a' \leq b' \leq b$  but  $a'/b' \neq a/b$ . We note a basic fact that if  $a/b \rightarrow c/d$  and  $(a, b) \in \theta$  for a congruence  $\theta \in \text{Con } L$  then also  $(c, d) \in \theta$ . The importance of the *weak projectivity of quotients of a lattice* in the description of lattice congruences is given by the following three results.

**Lemma 2.1.** ([2, Theorem III.1.2] or [6, Lemma 1]) *For any principal congruence  $\theta_{a,b} \in \text{Con } L$ ,*

$$(c, d) \in \theta_{a,b},$$

*( $d \leq c, b \leq a$ ) if and only if there is a finite chain  $d = y_0 \leq \dots \leq y_m = c$  such that  $a/b \rightarrow y_{i+1}/y_i$  for all  $i \in \{0, \dots, m-1\}$ .*

We use the following abbreviation. For a lattice  $L$  and  $a/b, c/d$  quotients of  $L$ , we shall be saying that *there are projections from the quotient  $a/b$  into a chain in  $c/d$*  or that  *$a/b$  has projections into a chain in  $c/d$*  if there exists a finite chain  $d = y_0 \leq \dots \leq y_m = c$  such that  $a/b \rightarrow y_{i+1}/y_i$  for all  $i \in \{0, \dots, m-1\}$ .

**Lemma 2.2.** ([11, 1.4] or [6, Lemma 2]) *Let  $L$  be a lattice and  $\theta, \varphi \in \text{Con } L$ . Then the relative pseudocomplement of  $\theta$  with respect to  $\varphi$  is*

$$\theta * \varphi = \bigvee (\theta_{u,v}, (u, v) \in S),$$

*where  $S$  is the set of all pairs of elements  $(u, v)$  ( $u, v \in L$ ) such that  $u/v \rightarrow z/t$  and  $(z, t) \in \theta$  implies  $(z, t) \in \varphi$  for all  $z, t \in L$ .*

**Corollary 2.3.** *Let  $L$  be a lattice,  $\theta, \varphi \in \text{Con } L$  and let  $a, b \in L$ ,  $b < a$ .*

- (i)  *$(a, b) \in \theta * \varphi$  if and only if for every projection  $a/b \rightarrow c/d$  ( $c, d \in L$ ),  $(c, d) \in \theta$  yields  $(c, d) \in \varphi$ ;*
- (ii)  *$(a, b) \notin \theta * \varphi$  if and only if there is a projection  $a/b \rightarrow c/d$  ( $c, d \in L$ ) such that  $(c, d) \in \theta$  and  $(c, d) \notin \varphi$ .*

*Proof.* It is sufficient to show (i). If  $(a, b) \in \theta * \varphi$  and  $a/b \rightarrow c/d$ , then  $(c, d) \in \theta * \varphi$ . Now  $(c, d) \in \theta$  and the definition of  $\theta * \varphi$  (see the beginning of Section 2) imply immediately that  $(c, d) \in \varphi$ . Conversely, let for every projection  $a/b \rightarrow c/d$  ( $c, d \in L$ ),  $(c, d) \in \theta$  yields  $(c, d) \in \varphi$ . Then  $(a, b) \in S$  for the set  $S$  from Lemma 2.2, hence  $(a, b) \in \theta * \varphi$ .  $\square$

All undefined terms can be found in G. Grätzer [2].

### 3. LATTICES WITH RELATIVE STONE CONGRUENCE LATTICES

In this section we give a new characterization of lattices with relative Stone congruence lattices which is alternative to that in [5].

We start with definitions, motivated by the symmetric identity (RS).

**Definition 3.1.** *Let  $L$  be a lattice. Let  $a/b, u/v, z/t$  be quotients of  $L$  and let  $\theta, \pi \in \text{Con } L$ . Then  $L$  is said to be **(RS)-weakly modular** whenever  $a/b \rightarrow u/v$  and  $a/b \rightarrow z/t$  with  $(u, v) \in \theta$  and  $(z, t) \in \pi$  imply that either one of (I)  $(u, v) \in \pi$ , (II)  $(z, t) \in \theta$  holds or the following condition is satisfied:*

- (III) *there are proper subquotients  $a_1/b_1 \subset a/b$ ,  $a_2/b_2 \subset a/b$  and quotients  $u'/v', z'/t'$  such that  $a_1/b_1 \rightarrow u'/v'$ ,  $(u', v') \in \theta$ ,  $(u', v') \notin \pi$  and also  $a_2/b_2 \rightarrow z'/t'$ ,  $(z', t') \in \pi$ ,  $(z', t') \notin \theta$ .*

**Definition 3.2.** *Let  $L$  be a lattice and let  $\theta, \pi$  be congruences of  $L$ . Then the pair  $\theta, \pi$  is said to be **(RS)-separable** if for every  $b < a$  in  $L$  there is a finite chain  $b = x_0 \leq \dots \leq x_m = a$  such that for every  $i \in \{0, \dots, m-1\}$  one of the following conditions holds:*

- (i) *for every proper subquotient  $r_1/s_1 \subset x_{i+1}/x_i$  and every projection  $r_1/s_1 \rightarrow u_1/v_1$ ,  $(u_1, v_1) \in \theta$  implies  $(u_1, v_1) \in \pi$ ;*
- (ii) *for every proper subquotient  $r_2/s_2 \subset x_{i+1}/x_i$  and every projection  $r_2/s_2 \rightarrow u_2/v_2$ ,  $(u_2, v_2) \in \pi$  implies  $(u_2, v_2) \in \theta$ .*

Now we can present our alternative characterization of lattices with relative Stone congruence lattices.

**Theorem 3.3.** *Let  $L$  be a lattice. Then  $\text{Con } L$  satisfies the identity*

$$(RS) \quad (\theta * \pi) \vee (\pi * \theta) = \nabla$$

*if and only if*

- (1)  $L$  is (RS)-weakly modular and
- (2) every pair  $\theta, \pi$  of congruences of  $L$  is (RS)-separable.

*Proof.* First we shall prove the necessity. Let  $\text{Con } L$  satisfy the identity (RS). We shall show that  $L$  is (RS)-weakly modular. All the quotients considered below are non-trivial. Let  $\theta, \pi \in \text{Con } L$  and let  $a/b \rightarrow u/v$  and  $a/b \rightarrow z/t$  with  $(u, v) \in \theta$ ,  $(z, t) \in \pi$ .

Since  $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$ , there exists a chain  $b = x_0 \leq \dots \leq x_m = a$  such that for each  $i \in \{0, \dots, m-1\}$ ,  $(x_{i+1}, x_i) \in \theta * \pi$  or  $(x_{i+1}, x_i) \in \pi * \theta$ . We shall distinguish the following cases.

(I) For every  $i \in \{0, \dots, m-1\}$ ,  $(x_{i+1}, x_i) \in \theta * \pi$ . Then  $(a, b) \in \theta * \pi$  and as  $a/b \rightarrow u/v$  and  $(u, v) \in \theta$ , by Corollary 2.3(i) we obtain  $(u, v) \in \pi$ . So the condition (I) of Definition 3.1 holds.

(II) For every  $i \in \{0, \dots, m-1\}$ ,  $(x_{i+1}, x_i) \in \pi * \theta$ . Then  $(a, b) \in \pi * \theta$  and analogously as above we get the condition (II) of Definition 3.1.

(III) Now assume that (I) and (II) do not hold so  $(x_{i+1}, x_i) \notin \theta * \pi$  and  $(x_{j+1}, x_j) \notin \pi * \theta$  for some  $i, j \in \{0, \dots, m-1\}$  and proper quotients  $x_{i+1}/x_i, x_{j+1}/x_j \subset a/b$ . Set  $a_1/b_1 := x_{i+1}/x_i$  and  $a_2/b_2 := x_{j+1}/x_j$ . Then  $(a_1, b_1) \notin \theta * \pi$  and  $(a_2, b_2) \notin \pi * \theta$  yield by Corollary 2.3(ii) that there exist projections  $a_1/b_1 \rightarrow u'/v'$ ,  $(u', v') \in \theta$ ,  $(u', v') \notin \pi$  and  $a_2/b_2 \rightarrow z'/t'$ ,  $(z', t') \in \pi$ ,  $(z', t') \notin \theta$ . We have proved (1).

Now let  $\theta, \pi \in \text{Con } L$  and  $a, b \in L$ ,  $b < a$ . As  $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$ , there exists a chain  $b = x_0 \leq \dots \leq x_m = a$  such that for each  $i \in \{0, \dots, m-1\}$ ,  $(x_{i+1}, x_i) \in \theta * \pi$  or  $(x_{i+1}, x_i) \in \pi * \theta$ . If for  $i \in \{0, \dots, m-1\}$  we have  $(x_{i+1}, x_i) \in \theta * \pi$ , then for every proper subquotient  $r_1/s_1 \subset x_{i+1}/x_i$  also  $(r_1, s_1) \in \theta * \pi$ , whence by Corollary 2.3(i) for every projection  $r_1/s_1 \rightarrow u_1/v_1$ ,  $(u_1, v_1) \in \theta$  implies  $(u_1, v_1) \in \pi$ . So the condition (i) of Definition 3.2 is satisfied. Analogously, if for  $i \in \{0, \dots, m-1\}$  we have  $(x_{i+1}, x_i) \in \pi * \theta$ , then for every proper subquotient  $r_2/s_2 \subset x_{i+1}/x_i$  and projection  $r_2/s_2 \rightarrow u_2/v_2$ ,  $(u_2, v_2) \in \pi$  implies  $(u_2, v_2) \in \theta$ , so (ii) of Definition 3.2 holds. This gives (2).

Now let (1) and (2) hold and  $\theta, \pi \in \text{Con } L$ ,  $a, b \in L$ ,  $b < a$ . By the (RS)-separability of  $\theta, \pi$ , there is a chain  $b = x_0 \leq \dots \leq x_m = a$  such that (i) or (ii) of Definition 3.2 holds for each  $i \in \{0, \dots, m-1\}$ . We distinguish two cases.

(a) For every  $i \in \{0, \dots, m-1\}$ ,  $(x_{i+1}, x_i) \in \theta * \pi$  or  $(x_{i+1}, x_i) \in \pi * \theta$ . Then  $(a, b) \in (\theta * \pi) \vee (\pi * \theta)$ . We show that the remaining case is impossible.

(b) Now assume that (a) above does not hold. So there is  $i \in \{0, \dots, m-1\}$  such that  $(x_{i+1}, x_i) \notin \theta * \pi$  and  $(x_{i+1}, x_i) \notin \pi * \theta$ . By Corollary 2.3(ii), there exist quotients  $u/v, z/t$  and projections  $x_{i+1}/x_i \rightarrow u/v$  with  $(u, v) \in \theta$ ,  $(u, v) \notin \pi$  and  $x_{i+1}/x_i \rightarrow z/t$  with  $(z, t) \in \pi$ ,  $(z, t) \notin \theta$ . Now we use that  $L$  is (RS)-weakly modular. If (I) of Definition 3.1 holds, then  $(u, v) \in \pi$ ,

a contradiction. Analogously, if (II) of Definition 3.1 holds, then  $(z, t) \in \theta$ , a contradiction.

Now let (III) of Definition 3.1 hold. So there are proper subquotients  $a_1/b_1 \subset x_{i+1}/x_i$ ,  $a_2/b_2 \subset x_{i+1}/x_i$  and projections  $a_1/b_1 \rightarrow u'/v'$ ,  $(u', v') \in \theta$ ,  $(u', v') \notin \pi$  and  $a_2/b_2 \rightarrow z'/t'$ ,  $(z', t') \in \pi$ ,  $(z', t') \notin \theta$ . If (i) of Definition 3.2 holds, then  $a_1/b_1 \rightarrow u'/v'$  and  $(u', v') \in \theta$  gives  $(u', v') \in \pi$ , a contradiction. Analogously, if (ii) of Definition 3.2 holds, then  $a_2/b_2 \rightarrow z'/t'$  and  $(z', t') \in \pi$  gives  $(z', t') \in \theta$ , a contradiction. The proof is complete.  $\square$

#### 4. LATTICES WHOSE RELATIVE STONE CONGRUENCE LATTICES SATISFY THE IDENTITY $(E_n)$

G. Grätzer and E.T. Schmidt in [3] (cf. also [2, Theorem III.4.9]) characterised lattices  $L$  whose congruence lattice is Boolean by introducing the concepts of a *weakly modular* lattice and of a *separable* congruence.

In this section we generalise their result within the subvarieties of all relative Stone algebras defined by the identities  $(E_n)$ ,  $n \geq 2$ . We recall that the variety of Boolean algebras is the subvariety defined by the identity  $(E_2)$ . The following definitions generalise the definitions of weakly modular lattice and of separable congruence introduced by G. Grätzer and E.T. Schmidt.

**Definition 4.1.** *Let  $L$  be a lattice and  $n \geq 2$ . Let  $a/b, u_1/v_1, \dots, u_n/v_n$  be quotients of  $L$  and  $\theta_1, \dots, \theta_{n+1} \in \text{Con } L$ . Then  $L$  is said to be  **$(\mathbf{E}_n)$ -modular** whenever*

$$a/b \rightarrow u_j/v_j \quad \text{and} \quad (u_j, v_j) \in \theta_j, \quad j = 1, \dots, n$$

*imply that either*

- (I<sub>j</sub>) *there is  $j \in \{1, \dots, n\}$  with  $(u_j, v_j) \in \theta_{j+1}$  or*
- (II) *for all  $j \in \{1, \dots, n\}$  there are proper subquotients  $a_j/b_j \subset a/b$  and projections  $a_j/b_j \rightarrow u'_j/v'_j$  with  $(u'_j, v'_j) \in \theta_j$  and  $(u'_j, v'_j) \notin \theta_{j+1}$ .*

**Definition 4.2.** *Let  $L$  be a lattice and  $n \geq 2$ . Then an (unordered) $(n+1)$ -tuple  $\theta_1, \dots, \theta_{n+1}$  is said to be  **$(\mathbf{E}_n)$ -separable** if for any  $b < a$  there exists a chain  $b = x_0 \leq x_1 \leq \dots \leq x_m = a$  such that for every  $i \in \{0, \dots, m-1\}$  there exists  $j \in \{1, \dots, n\}$  with the following property:*

- (j) *for every proper subquotient  $a'/b' \subset x_{i+1}/x_i$  and every quotient  $u'_j/v'_j$ ,  $a'/b' \rightarrow u'_j/v'_j$  and  $(u'_j, v'_j) \in \theta_j$  imply  $(u'_j, v'_j) \in \theta_{j+1}$ .*

The main result of this section is the following characterization of lattices  $L$  whose congruence lattice satisfies the identity  $(E_n)$ .

**Theorem 4.3.** *Let  $L$  be a lattice and  $n \geq 2$ . The congruence lattice  $\text{Con } L$  satisfies the identity  $(E_n)$  if and only if all of the following conditions hold:*

- (1)  $\text{Con } L$  is relative Stone;
- (2)  $L$  is  $(E_n)$ -modular;
- (3) every  $(n+1)$ -tuple of congruences  $\theta_1, \dots, \theta_{n+1}$  on  $L$  is  $(E_n)$ -separable.

*Proof.* Let  $L$  be a lattice and  $n \geq 2$ . We assume that  $\text{Con } L$  satisfies the identity

$$(E_n) \quad (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \dots \vee (\theta_n * \theta_{n+1}) = \nabla.$$

First we set  $\theta = \theta_1 = \theta_3 = \dots$  and  $\pi = \theta_2 = \theta_4 = \dots$  in the above congruence identity. Then we obtain that  $\text{Con } L$  satisfies the identity

$$(RS) \quad (\theta * \pi) \vee (\pi * \theta) = \nabla.$$

Hence  $\text{Con } L$  is relative Stone and (1) holds.

Now we prove (2). All the quotients considered below are non-trivial. Let  $\theta_1, \dots, \theta_{n+1} \in \text{Con } L$  and let  $a/b \rightarrow u_i/v_i$  with  $(u, v) \in \theta_i$  for  $i = 1, \dots, n$ .

Since  $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \dots \vee (\theta_n * \theta_{n+1})$ , there exists a chain  $b = x_0 \leq x_1 \leq \dots \leq x_m = a$  such that for every  $i \in \{0, \dots, m-1\}$  there exists  $j \in \{1, \dots, n\}$  with  $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$ .

We shall distinguish the following two cases.

(i) There is  $j \in \{1, \dots, n\}$  such that for every  $i \in \{0, \dots, m-1\}$  we have  $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$ . Then  $(a, b) \in \theta_j * \theta_{j+1}$  and since  $a/b \rightarrow u_j/v_j$  and  $(u_j, v_j) \in \theta_j$ , by Corollary 2.3(i) we obtain  $(u_j, v_j) \in \theta_{j+1}$ . So the condition  $(I_j)$  of Definition 3.1 holds.

(ii) Now assume that (i) above does not hold, so for all  $j \in \{1, \dots, n\}$  there is  $i_j \in \{0, \dots, m-1\}$  with  $(x_{i_j+1}, x_{i_j}) \notin \theta_j * \theta_{j+1}$ . Let us denote by  $a_j/b_j$  the proper quotient  $x_{i_j+1}/x_{i_j} \subset a/b$ ,  $j \in \{1, \dots, n\}$ . Then for each  $j \in \{1, \dots, n\}$ ,  $(a_j, b_j) \notin \theta_j * \theta_{j+1}$  yields by Corollary 2.3(ii) that there exists a projection  $a_j/b_j \rightarrow u'_j/v'_j$  with  $(u'_j, v'_j) \in \theta_j$  and  $(u'_j, v'_j) \notin \theta_{j+1}$ . So the condition (II) of Definition 3.1 holds. We have proved (1).

Now we will prove (2). Let  $\theta_1, \dots, \theta_{n+1}$  be congruences of the lattice  $L$ . Since  $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \dots \vee (\theta_n * \theta_{n+1})$ , there exists a chain  $b = x_0 \leq x_1 \leq \dots \leq x_m = a$  such that for every  $i \in \{0, \dots, m-1\}$  there exists  $j \in \{1, \dots, n\}$  with  $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$ .

Let  $i \in \{0, \dots, m-1\}$  and  $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$  for some  $j \in \{1, \dots, n\}$ . Let  $a'/b'$  be a proper subquotient of  $x_{i+1}/x_i$  and  $u'_j/v'_j$  be a quotient of  $L$  such that  $a'/b' \rightarrow u'_j/v'_j$  and  $(u'_j, v'_j) \in \theta_j$ . As  $(a', b') \in \theta_j * \theta_{j+1}$ , Corollary 2.3(ii) gives us  $(u'_j, v'_j) \in \theta_{j+1}$  as required. We have shown the necessity of (3).

Conversely, let the conditions (1)–(3) hold. Let  $\theta_1, \dots, \theta_{n+1} \in \text{Con } L$  and let  $b < a$  in  $L$ . By the  $(E_n)$ -separability of  $\theta_1, \dots, \theta_{n+1}$ , there exists a chain  $b = x_0 \leq x_1 \leq \dots \leq x_m = a$  such that for all  $i \in \{0, \dots, m-1\}$  there exists  $j \in \{1, \dots, n\}$  such that the condition (j) from Definition 4.2 is satisfied. We distinguish two cases.

(a) For every  $i \in \{0, \dots, m-1\}$  there exists some  $j \in \{1, \dots, n\}$  such that  $(x_{i+1}, x_i) \in \theta_j * \theta_{j+1}$ . Then  $(a, b) \in (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \dots \vee (\theta_n * \theta_{n+1})$  as required. We show that the remaining case is impossible.

(b) Assume that (a) above does not hold. So there exists  $i \in \{0, \dots, m-1\}$  such that for all  $j \in \{1, \dots, n\}$ ,  $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$ . By Corollary 2.3(ii), there exist quotients  $u_j/v_j$  and projections  $x_{i+1}/x_i \rightarrow u_j/v_j$  with  $(u_j, v_j) \in \theta_j$ ,  $(u_j, v_j) \notin \theta_{j+1}$ . Now we use that  $L$  is  $(E_n)$ -modular. If the condition  $(I_j)$  of Definition 4.1 holds, then  $(u_j, v_j) \in \theta_{j+1}$ , a contradiction.

Now let (II) of Definition 4.1 hold. So for all  $j \in \{1, \dots, n\}$  there are proper subquotients  $a_j/b_j \subset x_{i+1}/x_i$  and projections  $a_j/b_j \rightarrow u'_j/v'_j$  such that  $(u'_j, v'_j) \in \theta_j$ ,  $(u'_j, v'_j) \notin \theta_{j+1}$ . Since the condition (j) from Definition 4.2 is satisfied for certain  $j \in \{1, \dots, n\}$ ,  $a_j/b_j \rightarrow u'_j/v'_j$  and  $(u'_j, v'_j) \in \theta_j$  give  $(u'_j, v'_j) \in \theta_{j+1}$ , a contradiction. The proof is complete.  $\square$

As a corollary we get a new characterization of lattices with Boolean congruence lattices equivalent to the one by G. Grätzer and E.T. Schmidt (c.f.[3]).

**Corollary 4.4.** *Let  $L$  be a lattice.  $\text{Con } L$  is Boolean if and only if the following conditions hold:*

- (i)  $L$  is  $(E_2)$ -modular and
- (ii) every triple of congruences  $\theta_0, \theta_1, \theta_2$  on  $L$  is  $(E_2)$ -separable.

## 5. SEMI-DISCRETE CASES

Our main results, Theorem 4.3 and Theorem 3.3, can both be simplified if the lattice  $L$  is semi-discrete. Let us recall that a lattice  $L$  is called *semi-discrete* if between all comparable pairs of elements of  $L$  there exists a finite maximal chain. Every finite lattice is obviously semi-discrete.

**Corollary 5.1.** *Let  $L$  be a semi-discrete lattice. Then  $\text{Con } L$  is a relative Stone lattice satisfying the identity  $(E_n)$  ( $n \geq 2$ ) if and only if for any prime quotients  $p, q_1, \dots, q_{n-1}$  of  $L$  the projections  $p \rightarrow q_k$  for  $k \in \{1, \dots, n-1\}$  imply that one of the following conditions holds:*

- (i) *There exists  $j \in \{1, \dots, n-1\}$  such that there is a projection  $q_j \rightarrow p$ .*
- (ii) *There exist  $i, j \in \{1, \dots, n-1\}$ ,  $i \neq j$  such that there are projections  $q_i \rightarrow q_j$  and  $q_j \rightarrow q_i$ .*

*Proof.* Let  $L$  be a semi-discrete lattice and  $n \geq 2$ . We first prove the necessity. Let  $\text{Con } L$  satisfy the identity  $(E_n)$  and let  $p \rightarrow q_k$ ,  $k = 1, \dots, n-1$  for prime quotients  $p = a/b, q_1 = u_1/v_1, \dots, q_{n-1} = u_{n-1}/v_{n-1}$  of  $L$ . We set  $q_0 := a/b$  and  $\theta_0 := \theta_{a,b}$ ,  $\theta_1 := \theta_{u_1, v_1}$ ,  $\theta_{n-1} := \theta_{u_{n-1}, v_{n-1}}$ . We shall be using the abbreviations  $q_i \in \theta$  ( $q_i \notin \theta$ ) for  $(u_i, v_i) \in \theta$  ( $(u_i, v_i) \notin \theta$ ).

Firstly, by the (RS)-weakly modularity of  $L$ , for every  $i, j \in \{0, \dots, n-1\}$ ,  $i \neq j$ , the conditions  $p \rightarrow q_i$  and  $p \rightarrow q_j$  with  $q_i \in \theta_i$  and  $q_j \in \theta_j$  imply that

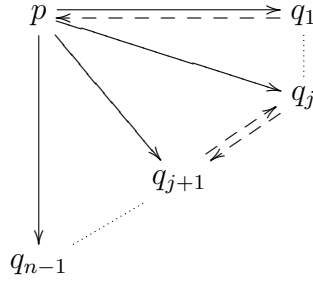
$q_i \in \theta_j$  or  $q_j \in \theta_i$  (we again note that the condition (III) in Definition 3.1 cannot be satisfied). So there exists an ordering  $i_0 \leq i_1 \leq \dots \leq i_{n-1}$  of the set  $\{0, \dots, n-1\}$  such that there are projections from the quotient  $q_{i_j}$  into the quotient  $q_{i_{j+1}}$  for  $j = 0, \dots, n-2$ . W.l.o.g. let us assume that  $i_j = j$  for  $j \in \{0, \dots, n-1\}$  so that  $q_0 \rightarrow q_1, q_1 \rightarrow q_2, \dots, q_{n-2} \rightarrow q_{n-1}$ .

Secondly, by the  $(E_n)$ -modularity of  $L$  we have that the condition  $(I_j)$  is satisfied (we again note that the condition (II) in Definition 4.1 cannot be satisfied). Hence there is  $j \in \{0, \dots, n-1\}$  with  $q_j \in \theta_{j+1}$ . If  $j = 0$  then we have  $(a, b) \in \theta_{u_1, v_1}$ , if  $j \in \{1, \dots, n-1\}$  then we have  $(u_j, v_j) \in \theta_{u_{j+1}, v_{j+1}}$ . Now Lemma 2.1 gives us for our prime quotients that  $u_1/v_1 \rightarrow a/b$  in case  $j = 0$  and  $u_{j+1}/v_{j+1} \rightarrow u_j/v_j$  in case  $j \in \{1, \dots, n-1\}$ . Hence  $q_1 \rightarrow p$  in the first case or  $q_j \rightarrow q_{j+1}$  and  $q_{j+1} \rightarrow q_j$  in the second case. We have proven the necessity.

For the converse, assume the given hypothesis, whence, by reordering the given quotients (if necessary), we can assume that for any prime quotients  $p, q_1, \dots, q_{n-1}$  of  $L$ , the projections  $p \rightarrow q_k$  for  $k \in \{1, \dots, n-1\}$  imply that the condition (i) in the hypothesis holds or the following condition (ii)' holds: there exists  $j \in \{1, \dots, n-2\}$  such that there are projections  $q_j \rightarrow q_{j+1}$  and  $q_{j+1} \rightarrow q_j$ .

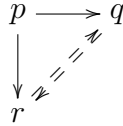
Suppose on the contrary that  $\text{Con } L$  does not satisfy the identity  $(E_n)$ , hence there are congruences  $\theta_1, \dots, \theta_{n+1} \in \text{Con } L$  and elements  $a, b \in L, a > b$ , such that  $(a, b) \notin (\theta_1 * \theta_2) \vee (\theta_2 * \theta_3) \vee \dots \vee (\theta_n * \theta_{n+1})$ . This means that for a maximal chain  $b = x_0 \leq x_1 \leq \dots \leq x_m = a$  between the elements  $a, b$  (such a chain exists as  $L$  is semi-discrete) we have that there is  $i \in \{0, \dots, m-1\}$  such that  $(x_{i+1}, x_i) \notin \theta_j * \theta_{j+1}$  for all  $j \in \{1, \dots, n\}$ . By Corollary 2.3(ii) this implies that for all  $j \in \{1, \dots, n\}$  there exist projections  $x_{i+1}/x_i \rightarrow u_j/v_j$  with  $(u_j, v_j) \in \theta_j, (u_j, v_j) \notin \theta_{j+1}$  for some elements  $u_j > v_j, j = 1, \dots, n$  in  $L$ . Since for every  $j$  there is a maximal chain  $v_j = y_{j_0} \leq y_{j_1} \leq \dots \leq y_{j_m} = u_j$  between the elements  $u_j, v_j$ , for every  $j$  there is a prime quotient  $y_{j_{i+1}}/y_{j_i}$  with  $(y_{j_i}, y_{j_{i+1}}) \in \theta_j$  and  $(y_{j_i}, y_{j_{i+1}}) \notin \theta_{j+1}$ . W.l.o.g. we can assume that the quotients  $q_j := u_j/v_j$  are already prime for  $j = 1, \dots, n$ . Of course, the quotient  $p := x_{i+1}/x_i$  is prime, too.

Now by the assumption we have that for our prime quotients  $p, q_2, \dots, q_n$ , the projections  $p \rightarrow q_2, \dots, p \rightarrow q_n$  imply one of the conditions (i), (ii)'. If (i) holds then there is  $j \in \{2, \dots, n\}$  such that  $q_j \rightarrow p$ ; since  $q_j \in \theta_j$ , we obtain  $p \in \theta_j$  and as  $j > 1$  and  $p \rightarrow q_{j-1}$ , we finally obtain  $q_{j-1} \in \theta_j$ , a contradiction. If (ii)' holds then there exists  $j \in \{1, \dots, n-2\}$  such that there are projections  $q_j \rightarrow q_{j+1}$  and  $q_{j+1} \rightarrow q_j$ . As  $q_{j+1} \in \theta_{j+1}$ , we obtain  $q_j \in \theta_{j+1}$ , a contradiction.  $\square$



Analogously, Theorem 3.3 yields the following statement.

**Corollary 5.2.** ([5], Corollary to Theorem 5) *Let  $L$  be a semi-discrete lattice. Then  $\text{Con } L$  is a relative Stone lattice if and only if for any prime quotients  $p, q, r$  of  $L$ , the projections  $p \rightarrow q$  and  $p \rightarrow r$  imply that  $q \rightarrow r$  or  $r \rightarrow q$ .*



## 6. CONCLUSION

In the presented work we give a new characterization of lattices with relative Stone congruence lattices. We also present the generalization, within the subvarieties of relative Stone Heyting algebras, of a characterization of G. Grätzer and E.T. Schmidt of lattices with Boolean congruence lattices.

As corollaries we obtain descriptions of semi-discrete lattices with relative Stone congruence lattices.

The presented results have already been accepted for publication in [4].

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